**PART – I**

# TYPICAL QUESTIONS & ANSWERS

## OBJECTIVE TYPE QUESTIONS

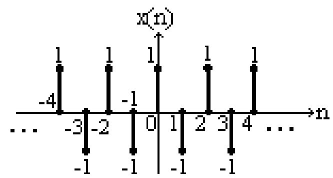
**Each Question carries 2 marks.**

**Choose the correct or best alternative in the following:**

**Q.1** The discrete-time signal $x(n) = (-1)^n$ is periodic with fundamental period

**(A)** 6 **(B)** 4
**(C)** 2 **(D)** 0

**Ans: C** Period = 2

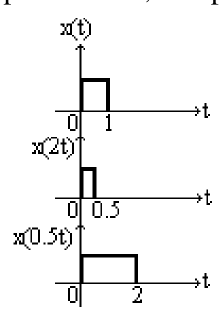



**Q.2** The frequency of a continuous time signal x (t) changes on transformation from x (t) to x ($\alpha$t), $\alpha > 0$ by a factor

**(A)** $\alpha$. **(B)** $\frac{1}{\alpha}$.
**(C)** $\alpha^2$. **(D)** $\sqrt{\alpha}$.

**Ans: A** $x(t) \xrightarrow{\text{Transform}} x(\alpha t), \alpha > 0$

$\alpha > 1 \Rightarrow$ compression in t, expansion in f by α.
$\alpha < 1 \Rightarrow$ expansion in t, compression in f by α.




**Q.3** A useful property of the unit impulse δ (t) is that

**(A)** $\delta(at) = a\,\delta(t)$. **(B)** $\delta(at) = \delta(t)$.
**(C)** $\delta(at) = \frac{1}{a}\delta(t)$. **(D)** $\delta(at) = [\delta(t)]^a$.

**Ans: C** Time-scaling property of δ(t):

$\delta(at) = \frac{1}{a}\delta(t), a > 0$

**Q.4** The continuous time version of the unit impulse $\delta(t)$ is defined by the pair of relations

**(A)** $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$. **(B)** $\delta(t) = 1,\ t = 0$ and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$.

**(C)** $\delta(t) = 0, t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$. **(D)** $\delta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$.

**Ans: C** $\delta(t) = 0, t \neq 0 \rightarrow \delta(t) \neq 0$ at origin

$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \rightarrow$ Total area under the curve is unity.

[$\delta(t)$ is also called Dirac-delta function]

**Q.5** Two sequences $x_1(n)$ and $x_2(n)$ are related by $x_2(n) = x_1(-n)$. In the z- domain, their ROC's are

**(A)** the same. **(B)** reciprocal of each other.
**(C)** negative of each other. **(D)** complements of each other.

.

**Ans: B** $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$, RoC $R_x$

$x_2(n) = x_1(-n) \overset{z}{\longleftrightarrow} X_1(1/z)$, RoC $1/R_x$

} Reciprocals

**Q.6** The Fourier transform of the exponential signal $e^{j\omega_0 t}$ is

**(A)** a constant. **(B)** a rectangular gate.
**(C)** an impulse. **(D)** a series of impulses.

**Ans: C** Since the signal contains only a high frequency $\omega_o$ its FT must be an impulse at $\omega = \omega_o$

**Q.7** If the Laplace transform of $f(t)$ is $\frac{\omega}{(s^2 + \omega^2)}$, then the value of $\lim_{t \to \infty} f(t)$

**(A)** cannot be determined. **(B)** is zero.
**(C)** is unity. **(D)** is infinity.

**Ans: B** $f(t) \overset{L}{\longleftrightarrow} \frac{\omega}{s^2 + \omega^2}$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s) \quad \text{[Final value theorem]}$$

$$= \lim_{s \to 0} \left( \frac{s\omega}{s^2 + \omega^2} \right) = 0$$

**Q.8** The unit impulse response of a linear time invariant system is the unit step function $u(t)$. For t > 0, the response of the system to an excitation

$e^{-at}u(t),\ a > 0,$ will be

**(A)** $ae^{-at}$. **(B)** $\frac{1-e^{-at}}{a}$.

**(C)** $a(1-e^{-at})$. **(D)** $1-e^{-at}$.

**Ans: B**

h(t) = u(t); x(t) = $e^{-at}$ u(t), a > 0

System response $y(t) = L^{-1}\left[\frac{1}{s}.\frac{1}{s+a}\right]$

$= L^{-1}\frac{1}{a}\left[\frac{1}{s}-\frac{1}{s+a}\right]$

$= \frac{1}{a}(1 - e^{-at})$

**Q.9** The z-transform of the function $\sum_{k=-\infty}^{0}\delta(n-k)$ has the following region of convergence

**(A)** $|z|>1$ **(B)** $|z|=1$

**(C)** $|z|<1$ **(D)** $0<|z|<1$

**Ans: C** $x(n) = \sum_{k=-\infty}^{0}\delta(n-k)$

$x(z) = \sum_{k=-\infty}^{0} z^{-k} = \ldots..+ z^3 + z^2 + z + 1$ (Sum of infinite geometric series)

$= \frac{1}{1-z}, \quad |z| < 1$

**Q.10** The auto-correlation function of a rectangular pulse of duration T is

**(A)** a rectangular pulse of duration T.
**(B)** a rectangular pulse of duration 2T.
**(C)** a triangular pulse of duration T.
**(D)** a triangular pulse of duration 2T.

**Ans: D**

$R_{XX}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, x(t+\tau)\, d\tau$ ⇨ triangular function of duration 2T.

**Q.11** The Fourier transform (FT) of a function x (t) is X (f). The FT of $dx(t)/dt$ will be

**(A)** $dX(f)/df$. **(B)** $j2\pi f\, X(f)$.
**(C)** $jf\, X(f)$. **(D)** $X(f)/(jf)$.

**Ans: B** $(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f)\, e^{j\omega t}\, d\omega$

$$\frac{dx}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega\, X(f)\, e^{j\omega t}\, d\omega$$

$$\therefore \frac{dx}{dt} \leftrightarrow j\, 2\pi\, f\, X(f)$$

**Q.12** The FT of a rectangular pulse existing between $t = -T/2$ to $t = T/2$ is a

**(A)** sinc squared function. **(B)** sinc function.
**(C)** sine squared function. **(D)** sine function.

**Ans: B** $x(t) = \begin{cases} 1, & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt = \left. \frac{e^{-j\omega t}}{j\omega} \right|_{-T/2}^{+T/2}$$

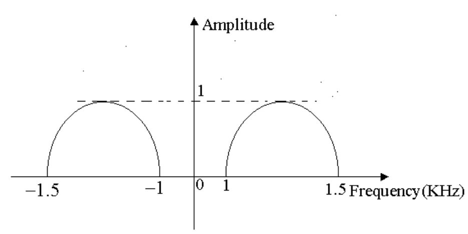
$$= -\frac{1}{j\omega}\left(e^{-j\omega T/2} - e^{j\omega T/2}\right) = \frac{2}{\omega}\left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right)$$

$$= \frac{2}{\omega} \sin\frac{\omega T}{2} = \frac{\sin(\omega T/2)}{\omega T/2} .T$$

Hence $X(j\omega)$ is expressed in terms of a sinc function.

**Q.13** An analog signal has the spectrum shown in Fig. The minimum sampling rate needed to completely represent this signal is

**(A)** 3 KHz .
**(B)** 2 KHz .
**(C)** 1 KHz .
**(D)** 0.5 KHz .




**Ans: C** For a band pass signal, the minimum sampling rate is twice the bandwidth, which is 0.5kHz here.

**Q.14** A given system is characterized by the differential equation:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t).$$

The system is :

**(A)** linear and unstable. **(B)** linear and stable.
**(C)** nonlinear and unstable. **(D)** nonlinear and stable.

**Ans:A** $\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$, $x(t) \rightarrow$ [ h(t) ] $\rightarrow y(t)$
system

The system is linear . Taking LT with zero initial conditions, we get
$s^2Y(s) - sY(s) - 2Y(s) = X(s)$

or, $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}$

Because of the pole at s = +2, the system is unstable.

**Q.15** The system characterized by the equation $y(t) = ax(t) + b$ is

**(A)** linear for any value of b. **(B)** linear if b > 0.
**(C)** linear if b < 0. **(D)** non-linear.

**Ans: D** The system is non-linear because x(t) = 0 does not lead to y (t) = 0, which is a violation of the principle of homogeneity.

**Q.16** Inverse Fourier transform of $u(\omega)$ is

**(A)** $\frac{1}{2}\delta(t) + \frac{1}{\pi t}$. **(B)** $\frac{1}{2}\delta(t)$.
**(C)** $2\delta(t) + \frac{1}{\pi t}$. **(D)** $\delta(t) + \mathrm{sgn}(t)$.

**Ans: A** $x(t) = u(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \pi\,\delta(\omega) + \frac{1}{J\omega}$

Duality property: $X(jt) \longleftrightarrow 2\pi\, x(-\omega)$

$u(\omega) \longleftrightarrow \frac{1}{2}\delta(t) + \frac{1}{\pi t}$

**Q.17** The impulse response of a system is $h(n) = a^n u(n)$. The condition for the system to be BIBO stable is

**(A)** a is real and positive. **(B)** a is real and negative.
**(C)** $|a| > 1$. **(D)** $|a| < 1$.

**Ans: D** Sum $S = \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} | a^n u(n) |$

$$\leq \sum_{n=0}^{+\infty} |a|^n \qquad (\because u(n) = 1 \text{ for } n \geq 0)$$

$$\leq \frac{1}{1-|a|} \qquad \text{if } |a| < 1.$$

**Q.18** If $R_1$ is the region of convergence of x (n) and $R_2$ is the region of convergence of y(n), then the region of convergence of x (n) convoluted y (n) is

**(A)** $R_1+R_2$. **(B)** $R_1-R_2$.
**(C)** $R_1 \cap R_2$. **(D)** $R_1 \cup R_2$.

**Ans:C** $x(n) \overset{z}{\longleftrightarrow} X(z)$, RoC $R_1$

$y(n) \overset{z}{\longleftrightarrow} Y(z)$, RoC $R_2$

$x(n) * y(n) \overset{z}{\longleftrightarrow} X(z).Y(z)$, RoC at least $R_1 \cap R_2$

**Q.19** The continuous time system described by $y(t) = x(t^2)$ is
**(A)** causal, linear and time varying.
**(B)** causal, non-linear and time varying.
**(C)** non causal, non-linear and time-invariant.
**(D)** non causal, linear and time-invariant.

**Ans: D**

$y(t) = x(t^2)$

y(t) depends on $x(t^2)$ i.e., future values of input if $t > 1$.

$\therefore$ System is anticipative or <u>non-causal</u>

$\alpha x_1(t) \rightarrow y_1(t) = \alpha x_1(t^2)$

$\beta x_2(t) \rightarrow y_2(t) = \beta x_2(t^2)$

$\therefore \alpha x_1(t) + \beta x_2(t) \rightarrow y(t) = \alpha x_1(t^2) + \beta x_2(t^2) = y_1(t) + y_2(t)$

$\therefore$ System is <u>Linear</u>

System is time varying. Check with $x(t) = u(t) - u(t\text{-}z) \rightarrow y(t)$ and

$x_1(t) = x(t-1) \rightarrow y_1(t)$ and find that $y_1(t) \neq y(t-1)$.

**Q.20** If G(f) represents the Fourier Transform of a signal g (t) which is real and odd symmetric in time, then G (f) is

**(A)** complex. **(B)** imaginary.
**(C)** real. **(D)** real and non-negative.

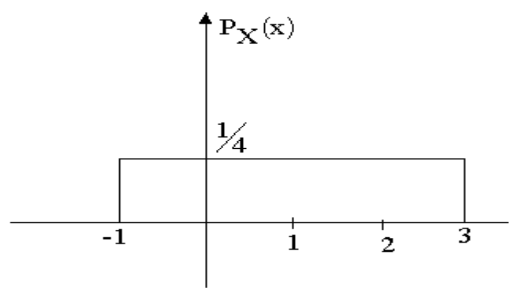
**Ans:B** $g(t) \overset{FT}{\longleftrightarrow} G(f)$

g(t) real, odd symmetric in time

$G^*(j\omega) = - G(j\omega)$; $G(j\omega)$ purely imaginary.

**Q.21** For a random variable x having the PDF shown in the Fig., the mean and the variance are, respectively,

**(A)** $\frac{1}{2}$ and $\frac{2}{3}$.
**(B)** 1 and $\frac{4}{3}$.
**(C)** 1 and $\frac{2}{3}$.
**(D)** 2 and $\frac{4}{3}$.




**Ans:B** Mean $= \mu_x(t) = \int_{-\infty}^{+\infty} x\, f_{x(t)}(x)\, dx$

$$= \int_{-1}^{3} x \frac{1}{4} dx = \frac{1}{4} \frac{x^2}{2} \Big|_{-1}^{3} = \left(\frac{9}{2} - \frac{1}{2}\right)\frac{1}{4} = 1$$

Variance $= \int_{-\infty}^{+\infty} (x - \mu_x)^2 f_x(x)\, dx$

$$= \int_{-1}^{3} (x - 1)^2 \frac{1}{4} d(x-1)$$

$$= \frac{1}{4} \frac{(x - 1)^3}{3} \Big|_{-1}^{3} = \frac{1}{12}[8 + 8] = \frac{4}{3}$$

**Q.22** If white noise is input to an RC integrator the ACF at the output is proportional to

**(A)** $\exp\left(\frac{-|\tau|}{RC}\right)$. **(B)** $\exp\left(\frac{-\tau}{RC}\right)$.

**(C)** $\exp(|\tau| RC)$. **(D)** $\exp(-\tau RC)$.

**Ans: A**

$$R_N(\tau) = \frac{N_0}{4RC}\left(\exp -\frac{|\tau|}{RC}\right)$$

**Q.23** $x(n) = a^{|n|}, |a| < 1$ is

**(A)** an energy signal.
**(B)** a power signal.
**(C)** neither an energy nor a power signal.
**(D)** an energy as well as a power signal.

**Ans: A**

$$\text{Energy} = \sum_{n=-\infty}^{+\infty} x^2(n) = \sum_{n=-\infty}^{\infty} a^{2|n|} = \sum_{n=-\infty}^{\infty} (a^2)^{|n|} = 1 + 2\sum_{n=1}^{\infty} a^2$$

$= \text{finite since } |a| < 1$

$\therefore$ This is an energy signal.

**Q.24** The spectrum of x (n) extends from $-\omega_o$ to $+\omega_o$, while that of h(n) extends from $-2\omega_o$ to $+2\omega_o$. The spectrum of $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$ extends from

**(A)** $-4\omega_o$ to $+4\omega_o$. **(B)** $-3\omega_o$ to $+3\omega_o$.

**(C)** $-2\omega_o$ to $+2\omega_o$. **(D)** $-\omega_o$ to $+\omega_o$

**Ans: D** Spectrum depends on $H(e^{j\omega}) \longrightarrow X(e^{j\omega})$ Smaller of the two ranges.

**Q.25** The signals $x_1(t)$ and $x_2(t)$ are both bandlimited to $(-\omega_1, +\omega_1)$ and $(-\omega_2, +\omega_2)$ respectively. The Nyquist sampling rate for the signal $x_1(t)x_2(t)$ will be

**(A)** $2\omega_1$ if $\omega_1 > \omega_2$. **(B)** $2\omega_2$ if $\omega_1 < \omega_2$.

**(C)** $2(\omega_1+\omega_2)$. **(D)** $(\omega_1+\omega_2)/2$.

**Ans: C** Nyquist sampling rate = 2(Bandwidth) = $2(\omega_1 - (-\omega_2)) = 2(\omega_1 + \omega_2)$

**Q.26** If a periodic function f(t) of period T satisfies $f(t) = -f\left(t + T/2\right)$, then in its Fourier series expansion,

**(A)** the constant term will be zero.
**(B)** there will be no cosine terms.
**( C)** there will be no sine terms.
**(D)** there will be no even harmonics.

**Ans:**

$$\frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\left(\int_0^{T/2} f(t)\,dt + \int_{T/2}^{T} f(t)\,dt\right) = \frac{1}{T}\left(\int_0^{T/2} f(t)\,dt + \int_0^{T/2} f(\tau + T/2)\,d\tau\right) = 0$$

**Q.27** A band pass signal extends from 1 KHz to 2 KHz. The minimum sampling frequency needed to retain all information in the sampled signal is

**(A)** 1 KHz. **(B)** 2 KHz.
**(C)** 3 KHz. **(D)** 4 KHz.

**Ans: B**

Minimum sampling frequency = 2(Bandwidth) = 2(1) = 2 kHz

**Q.28** The region of convergence of the z-transform of the signal

$$2^n u(n) - 3^n u(-n-1)$$

**(A)** is $|z| > 1$. **(B)** is $|z| < 1$.
**(C)** is $2 < |z| < 3$. **(D)** does not exist.

**Ans:**

$$2^n u(n) \longleftrightarrow \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$$3^n u(-n-1) \longleftrightarrow \frac{1}{1-3z^{-1}}, \quad |z| < 3$$

$\therefore$ ROC is $2 < |z| < 3$.

**Q.29** The number of possible regions of convergence of the function $\dfrac{\left(e^{-2}-2\right)z}{\left(z-e^{-2}\right)\left(z-2\right)}$ is

**(A)** 1. **(B)** 2.
**(C)** 3. **(D)** 4.

**Ans: C**

Possible ROC's are $|z| > e^{-2}$, $|z| < 2$ and $e^{-2} < |z| < 2$

**Q.30** The Laplace transform of u(t) is A(s) and the Fourier transform of u(t) is $B(j\omega)$. Then

**(A)** $B(j\omega) = A(s)\big|_{s=j\omega}$. **(B)** $A(s) = \frac{1}{s}$ but $B(j\omega) \neq \frac{1}{j\omega}$.

**(C)** $A(s) \neq \frac{1}{s}$ but $B(j\omega) = \frac{1}{j\omega}$. **(D)** $A(s) \neq \frac{1}{s}$ but $B(j\omega) \neq \frac{1}{j\omega}$.

**Ans: B** $u(t) \overset{L}{\Longleftrightarrow} A(s) = \frac{1}{s}$

$u(t) \overset{F.T}{\Longleftrightarrow} B(j\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)$

$\therefore A(s) = \frac{1}{s}$ but $B(j\omega) \neq \frac{1}{j\omega}$

**PART – II**

# NUMERICALS & DERIVATIONS

**Q.1.** Determine whether the system having input x (n) and output y (n) and described by relationship : $y(n) = \sum_{k=-\infty}^{n} x(k+2)$

is (i) memoryless, (ii) stable, (iii)causal (iv) linear and (v) time invariant. **(5)**

**Ans:**

$$y(n) = \sum_{k=-\infty} x(k+2)$$

(i) Not memoryless - as y(n) depends on past values of input from x(-∞) to x(n-1) (assuming)n > 0)

(ii) Unstable- since if |x (n)| ≤ M, then |y(n)| goes to ∞ for any n.

(iii) Non-causal - as y(n) depends on x(n+1) as well as x(n+2).

(iv) Linear - ∵ the principle of superposition applies (due to ∑ operation)

(v) Time – invariant - ∵ a time-shift in input results in corresponding time-shift in output.

**Q.2.** Determine whether the signal x (t) described by x (t) = exp [- at] u (t), a > 0 is a power signal or energy signal or neither. **(5)**

**Ans:**

$x(t) = e^{-at} u(t), a > 0$

x(t) is a non-periodic signal.

$$\text{Energy } E = \int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{0}^{\infty} e^{-2at} dt = \frac{e^{-2at}}{-2a}\Bigg|_0^{\infty} = \frac{1}{2a} \text{ (finite, positive)}$$

The energy is finite and deterministic.

∴ x(t) is an energy signal.

**Q.3.** Determine the even and odd parts of the signal x (t) given by

$$x(t) = \begin{cases} A e^{-\alpha t} & t > 0 \\ 0 & t < 0 \end{cases}$$

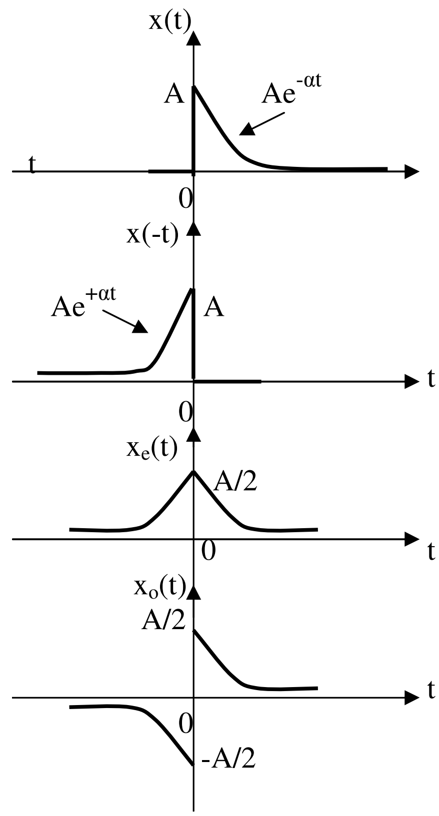
$$x(t) = \begin{cases} Ae^{-\alpha t} & t > 0 \\ 0 & t < 0 \end{cases}$$ **(5)**

**Ans:**

Assumption : α > 0, A > 0, -∞ < t < ∞

Even part $x_e(t) = \frac{x(t) + x(-t)}{2}$

Odd part $x_o(t) = \frac{x(t) - x(-t)}{2}$




**Q.4.** Use one sided Laplace transform to determine the output y (t) of a system described by

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y\,(t)=0 \text{ where } y(0-)=3 \text{ and } \left.\frac{dy}{dt}\right|_{t=0-}=1 \qquad \textbf{(7)}$$

**Ans:**

$$\frac{d^2y}{dt^2} + 3\,\frac{dy}{dt} + 2\,y(t) = 0,\quad y(0-) = 3,\quad \left.\frac{dy}{dt}\right|_{t=0^-} = 1$$

$$\left(s^2\,Y(s) - s\,y(0) - \left.\frac{dy}{dt}\right|_{t=0}\right) + 3\,[s\,Y(s) - y(0)] + 2\,Y(s) = 0$$

$$(s^2 + 3s + 2)\,Y(s) = sy(0) + \left.\frac{dy}{dt}\right|_{t=0} + 3\,y(0)$$

$$(s^2 + 3s + 2)\,Y(s) = 3s + 1 + 9 = 3s + 10$$

$$Y(s) = \frac{3s + 10}{s^2 + 3s + 2} = \frac{3s + 10}{(s + 1)(s + 2)}$$

$$= \frac{A}{s + 1} + \frac{B}{s + 2}$$

$A = \left.\dfrac{3s + 10}{s + 2}\right|_{s=-1} = 7$ ; $B = \left.\dfrac{3s + 10}{s + 1}\right|_{s=-2} = -4$

$\therefore Y(s) = \dfrac{7}{s+1} - \dfrac{4}{s+2}$

$\therefore y(t) = L^{-1}[Y(s)] = 7e^{-t} - 4e^{-2t} = e^{-t}(7 - 4e^{-t})$

$\therefore$ The output of the system is $y(t) = e^{-t}(7 - 4e^{-t})\, u(t)$

**Q. 5.** Obtain two different realizations of the system given by $y(n) - (a+b)\, y(n-1) + ab\, y(n-2) = x(n)$.Also obtain its transfer function. **(7)**

**Ans:**

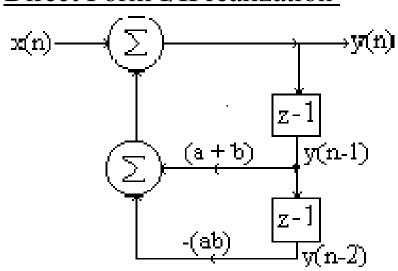
$y(n) - (a + b)\, y(n-1) + ab\, y(n-2) = x(n)$

$\therefore Y(z) - (a+b)\, z^{-1} Y(z) + ab\, z^{-2} Y(z) = X(z)$

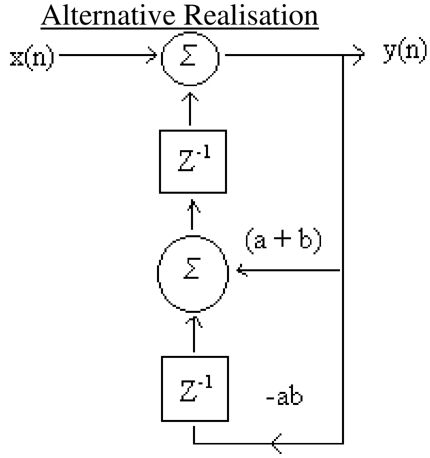
Transfer function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1 - (a+b)\, z^{-1} + ab\, z^{-2}}$

$y(n) = x(n) + (a + b)\, y(n-1) - ab\, y(n-2)$

Direct Form I/II realization




Alternative Realisation




**Q. 6.** An LTI system has an impulse response $h(t) = \exp[-at]\, u(t)$; when it is excited by an input signal x (t), its output is $y(t) = [\exp(-bt) - \exp(-ct)]\, u(t)$ Determine its input x (t). **(7)**

**Ans:**

$h(t) = e^{-at} u(t)$ for input x(t)

Output $y(t) = (e^{-bt} - e^{-ct})\, u(t)$

$h(t) \overset{L}{\longleftrightarrow} H(s)$, $y(t) \overset{L}{\longleftrightarrow} Y(s)$, $x(t) \overset{L}{\longleftrightarrow} X(s)$

$H(s) = \dfrac{1}{s+a}$ ; $Y(s) = \dfrac{1}{s+b} - \dfrac{1}{s+c} = \dfrac{s + c - s - b}{(s + b)(s + c)} = \dfrac{c - b}{(s + b)(s + c)}$

As $H(s) = \frac{Y(s)}{X(s)}$ , $X(s) = \frac{Y(s)}{H(s)}$

$$\therefore X(s) = \frac{(c - b)(s + a)}{(s + b)(s + c)} = \frac{A}{s + b} + \frac{B}{s + c}$$

$$A = \left.\frac{(c - b)(s + a)}{(s + c)}\right|_{s = -b} = \frac{(c - b)(-b + a)}{(-b + c)} = a - b$$

$$B = \left.\frac{(c - b)(s + a)}{(s + b)}\right|_{s = -c} = \frac{(c - b)(-c + a)}{(-c + b)} = c - a$$

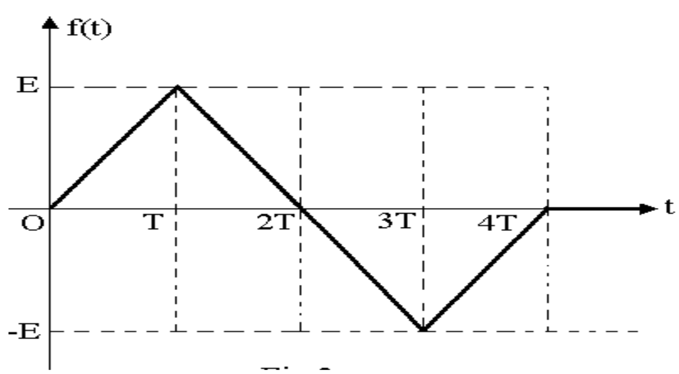
$$\therefore X(s) = \frac{a - b}{s + b} + \frac{c - a}{s + c}$$

$$x(t) = (a - b)\, e^{-bt} + (c - a)\, e^{-ct}$$

$\therefore$ The input $x(t) = [(a - b)\, e^{-bt} + (c - a)\, e^{-ct}]\, u(t)$

**Q.7.** Write an expression for the waveform $f(t)$ shown in Fig. using only unit step function and powers of t. **(3)**

**Ans:**

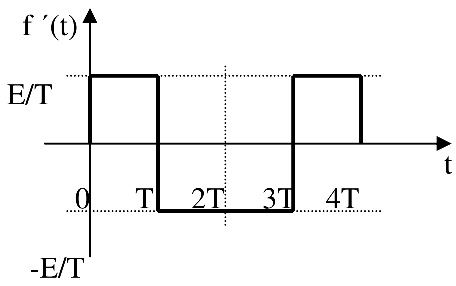



$$\therefore f(t) = \frac{E}{T}\,[\, t\, u(t) - 2(t - T)\, u(t - T) + 2(t - 3T)\, u(t - 3T) - (t - 4T)\, u(t - 4T)]$$

**Q.8.** For f(t) of Q7, find and sketch $f'(t)$ (prime denotes differentiation with respect to t). **(3)**

**Ans:**

$$f(t) = \frac{E}{T}\left[\, t\, u(t) - 2(t-T)\, u(t-T) + 2(t-3T)\, u(t-3T) - (t-4T)\, u(t-4T)\right]$$




$$\therefore f'(t) = \frac{E}{T}\left[u(t) - 2\, u(t-T) + 2\, u(t-3T) - u(t-4T)\right]$$

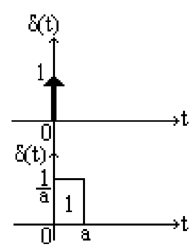
**Q.9.** Define a unit impulse function $\delta(t)$. **(2)**

**Ans:**

. Unit impulse function δ(t) is defined as:

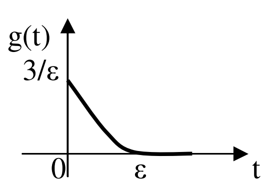
$$\begin{cases} \delta(t) = 0, \; t \neq 0 \\ \int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \end{cases}$$

It can be viewed as the limit of a rectangular pulse of duration a and height 1/a when a ⟶ 0, as shown below.




**Q.10.** Sketch the function $g(t) = \frac{3}{\epsilon^3}(t-\epsilon)^2[u(t) - u(t-\epsilon)]$ and show that $g(t) \to \delta(t)$ as $\epsilon \to 0$. **(6)**

**Ans:**




As ε ⟶ 0, duration ⟶ 0, amplitude ⟶ ∞

$$\int_0^{\varepsilon} g(t)\, dt = 1$$

**Q.11.** Show that if the FT of x (t) is $X(j\omega)$, then the FT of $x\left(\frac{t}{a}\right)$ is $|a| X(ja\omega)$. **(6)**

**Ans:**

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$$

Let $x\left[\frac{t}{a}\right] \overset{FT}{\longleftrightarrow} X_1(j\omega)$ , then

$$X_1(j\omega) = \int_{-\infty}^{+\infty} x\left[\frac{t}{a}\right] e^{-j\omega t}\, dt \qquad \text{Let } \frac{t}{a} = \alpha \quad \therefore dt = a\, d\alpha$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j\omega a\alpha}\, a\, d\alpha \text{ if } a > 0$$

$$- \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j\omega a\alpha}\, a\, d\alpha \text{ if } a < 0$$

Hence $X_1(j\omega) = |a| \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j\omega a\alpha}\, d\alpha = |a|\, x\,(j\omega a)$

**Q.12.** Solve, by using Laplace transforms, the following set of simultaneous differential equations for x (t). **(14)**

**Ans:**

$$2x'(t) + 4x(t) + y'(t) + 7y(t) = 5u(t)$$
$$x'(t) + x(t) + y'(t) + 3y(t) = 5\delta(t)$$

The initial conditions are : $x(0-) = y(0-) = 0$.

$$2\, x'(t) + 4\, x(t) + y'(t) + 7\, y(t) = 5\, u(t)$$

$$x'(t) + x(t) + y'(t) + 3\, y(t) = 5\, \delta(t)$$

$$x(t) \overset{L}{\longleftrightarrow} X(s),\ x'(t) \overset{L}{\longleftrightarrow} s\,X(s),\ \delta(t) \overset{L}{\longleftrightarrow} 1,\ u(t) \overset{L}{\longleftrightarrow} \frac{1}{s}$$

(Given zero initial conditions)

$$\therefore 2\, sX(s) + 4\, X(s) + sY(s) + 7\, Y(s) = \frac{5}{s}$$

$$sX(s) + X(s) + sY(s) + 3\, Y(s) = 5$$

$$(2s + 4)\, X(s) + (s+7)\, Y(s) = \frac{5}{s}$$

$$(s + 1)\, X(s) + (s+3)\, Y(s) = 5$$

$$X(s) = \frac{\begin{vmatrix} \frac{5}{S} & s+7 \\ 3 & \\ 5 & s+3 \end{vmatrix}}{\begin{vmatrix} 2s+4 & s+7 \\ s+1 & s+3 \end{vmatrix}}$$

Or, $X(s) = -\dfrac{5s + 35 - 5 - 15/s}{2s^2 + 6s + 4s + 12 - s^2 - 8s - 7}$

$$= -\frac{5s^2 + 30s - 15}{s(s^2 + 2s + 5)} = -\frac{5}{s}\left(\frac{s^2 + 6s - 3}{s^2 + 2s + 5}\right) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

Then $A(s^2 + 2s + 5) + Bs^2 + Cs = -5(s^2 + 6s - 3)$

$\therefore$ $A + B = -5$
$2A + C = -30$
$5A = 15$

Thus $A = 3$, $B = -8$, $C = -36$ and we can write

$$X(s) = \frac{3}{s} - 8\frac{s+1}{(s+1)^2 + 2^2} - 14\frac{2}{(s+1)^2 + 2^2}$$

$\therefore x(t) = (3 - 8e^{-t}\cos 2t - 14e^{-t}\sin 2t)\,u(t)$

**Q.13.** Find the Laplace transform of $t \sin \omega_0 t\, u(t)$. **(6)**

**Ans:**

$$\sin(\omega_0 t) \overset{L}{\longleftrightarrow} \frac{\omega_0}{s^2 + \omega_0^2}$$

Using $t f(t) \overset{L}{\longleftrightarrow} -\dfrac{d}{ds}[F(s)]$,

$$L[t \sin(\omega_0 t) u(t)] = -\frac{d}{ds}\left[\frac{\omega_0}{s^2 + \omega_0^2}\right]$$

$$= \left[\frac{0 - \omega_0(2s)}{(s^2 + \omega_0^2)^2}\right] = \frac{2\omega_0 s}{(s^2 + \omega_0^2)^2}$$

**Q.14.** Find the inverse Laplace transform of $\dfrac{s-2}{s(s+1)^3}$. **(8)**

**Ans:**

$$F(s) = \frac{s-2}{s(s+1)^3} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$A = \left.\dfrac{s-2}{(s+1)^3}\right|_{s=0} = -2$

$D = \left.\dfrac{s-2}{s}\right|_{s=-1} = 3$

$A = -2$ $D = 3$

$A(s+1)^3 + Bs(s+1)^2 + Cs(s+1) + Ds = s-2$

$s^3 : A + B = 0$ $B = 2$

$s^2 : 3A + 2B + C = 0$ $C = 2$

$$F(s) = \frac{-2}{s} + \frac{2}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{(s+1)^3}$$

$$\therefore f(t) = -2 + 2\,e^{-t} + 2\,t\,e^{-t} + \frac{3}{2}t^2 e^{-t}$$

$$\therefore f(t) = \left[-2 + e^{-t}\left(\frac{3}{2}t^2 + 2t + 2\right)\right] u(t)$$

**Q.15.** Show that the difference equation $y(n) - \alpha y(n-1) = -\alpha x(n) + x(n-1)$ represents an all-pass transfer function. What is (are) the condition(s) on $\alpha$ for the system to be stable? **(8)**

**Ans:**

$y(n) - \alpha\, y(n-1) = -\alpha\, x(n) + x(n-1)$

$Y(z) - \alpha\, z^{-1}\, Y(z) = -\alpha\, X(z) + z^{-1}\, X(z)$

$(1 - \alpha\, z^{-1})\, Y(z) = (-\alpha + z^{-1})\, X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\alpha + z^{-1}}{1 - \alpha\, z^{-1}} = \frac{1 - \alpha\, z}{z - \alpha}$$

Zero : $z = \frac{1}{\alpha}$

Pole : $z = \alpha$

As poles and zeros have reciprocal values, the transfer function represents an all pass filter system.

<u>Condition for stability of the system</u> :

For stability, the pole at $z = \alpha$ must be inside the unit circle, i.e. $|\alpha| < 1$.

**Q.16.** Give a recursive realization of the transfer function $H(z) = 1 + z^{-1} + z^{-2} + z^{-3}$ **(6)**

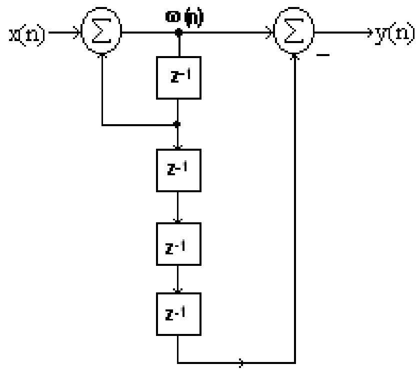
**Ans:**

$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} = \frac{1 - z^{-4}}{1 - z^{-1}}$$ (Geometric series of 4 terms; First term = 1, Common ratio = $z^{-1}$)

As $H(z) = \frac{Y(z)}{X(z)}$, we can write

$$\therefore (1 - z^{-1})\, Y(z) = (1 - z^{-4})\, X(z) \text{ or } Y(z) = \frac{X(z)}{(1 - z^{-1})}(1 - z^{-4}) = W(z)(1 - z^{-4})$$

The realization of the system is shown below.




**Q.17** Determine the z-transform of $x_1(n) = \alpha^n u(n)$ and $x_2(n) = -\alpha^n u(-n-1)$ and indicate their regions of convergence. **(6)**

**Ans:**

$x_1(n) = \alpha^n u(n)$ and $x_2(n) = -\alpha^n u(-n-1)$

$X_1(z) = \frac{1}{1-\alpha z^{-1}}$ RoC $|\alpha z^{-1}| < 1$ i.e., $|z| > \alpha$

$$X_2(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \alpha^{-n} z^n = -(\alpha^{-1}z + \alpha^{-2}z^2 + \alpha^{-3}z^3 + \ldots\ldots\ldots)$$

$$= -\alpha^{-1}z\,(1 + \alpha^{-1}z + \alpha^{-2}z^2 + \ldots\ldots..)$$

$$= \frac{-\alpha^{-1}z}{1-\alpha^{-1}z} = \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}\,; \quad \text{RoC } |\alpha^{-1} z| < 1 \text{ i.e., } |z| < |\alpha|$$

**Q.18.** Determine the sequence $h(n)$ whose z-transform is

$$H(z) = \frac{1}{1 - 2\,r\cos\theta\, z^{-1} + r^2 z^{-2}}, \quad |r| < 1.$$ **(6)**

**Ans:**

.

$$H(z) = \frac{1}{1-2r\cos\theta\, z^{-1} + r^2 z^{-2}}, \quad |r| < 1$$

$$= \frac{1}{(1-r e^{j\theta} z^{-1})(1-r e^{-j\theta} z^{-1})}, \quad |r| < 1$$

$$= \frac{A}{(1-r e^{j\theta} z^{-1})} + \frac{B}{(1-r e^{-j\theta} z^{-1})} = |r| < 1$$

where $A = \left.\dfrac{1}{(1-r e^{j\theta} z^{-1})}\right|_{r e^{j\theta} z^{-1}=1} = \dfrac{1}{1-e^{-j2\theta}}$

$B = \left.\dfrac{1}{(1-r e^{j\theta} z^{-1})}\right|_{r e^{-j\theta} z^{-1}=1} = \dfrac{1}{1-e^{j2\theta}}$

$\therefore h(n) = \dfrac{1}{1-e^{-2j\theta}} (r e^{j\theta})^n + \dfrac{1}{1-e^{2j\theta}} (r e^{-j\theta})^n$

$\therefore h(n) = r^n \left[\dfrac{e^{jn\theta}}{1-e^{-j2\theta}} + \dfrac{e^{-jn\theta}}{1-e^{j2\theta}}\right] u(n)$

$= r^n \dfrac{e^{j(n+1)\theta} - e^{-j(n+1)\theta}}{e^{j\theta} - e^{-j\theta}} \; u(n)$

$= \dfrac{r^n \sin(n+1)\theta}{\sin\theta} \; u(n)$

**Q.19.** Let the Z- transform of x(n) be X(z).Show that the z-transform of x (-n) is $X\left(\frac{1}{z}\right)$. **(2)**

**Ans:**

$x(n) \overset{z}{\longleftrightarrow} X(z)$ Let $y(n) = x(-n)$

Then $Y(z) = \sum_{n=-\infty}^{\infty} x(-n) z^{-n} = \sum_{r=-\infty}^{\infty} x(r) z^{+r} = \sum_{r=-\infty}^{\infty} x(r) (z^{-1})^{-1} = X(z^{-1})$

**Q.20.** Find the energy content in the signal $x(n) = e^{-n/10} \sin\left(\frac{2\pi n}{4}\right)$. **(7)**

**Ans:**

$x(n) = e^{-0.1n} \sin\left(\frac{2\pi n}{4}\right)$

Energy content $E = \sum_{n=-\infty}^{+\infty} |x^2(n)| = \sum_{n=-\infty}^{+\infty} e^{-0.2n} \left(\sin\left(\frac{2\pi n}{4}\right)\right)^2$

$E = \sum_{n=-\infty}^{+\infty} e^{-2n} \sin^2 \frac{n\pi}{2}$

$E = \sum_{n=-\infty}^{+\infty} e^{-2n} \frac{1-\cos n\pi}{2}$

$= \frac{1}{2} \sum_{n=-\infty}^{+\infty} e^{-2n} [1-(-1)^n]$

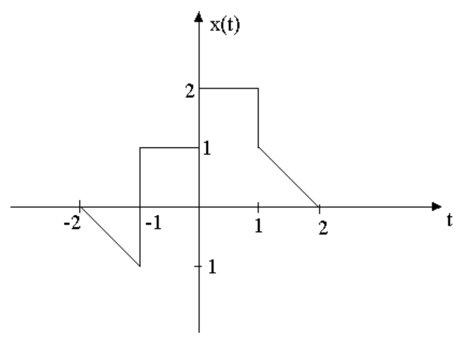
Now $1 - (-1)^n = \begin{cases} 2 \text{ for n odd} \\ 0 \text{ for n even} \end{cases}$

Also Let n = 2r +1 ; then $E = \sum_{r=-\infty}^{\infty} e^{-.2(2r+1)} = \sum_{r=-\infty}^{\infty} e^{-.4r} \; e^{-.2}$

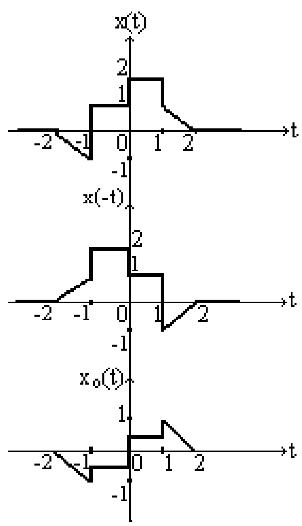
$= e^{-.2}\left[\sum_{r=0}^{\infty} e^{-.4r} + \sum_{r=1}^{\infty} e^{.4r}\right]$ The second term in brackets goes to infinity . Hence E is infinite.

**Q.21.** Sketch the odd part of the signal shown in Fig. **(3)**

**Ans:**

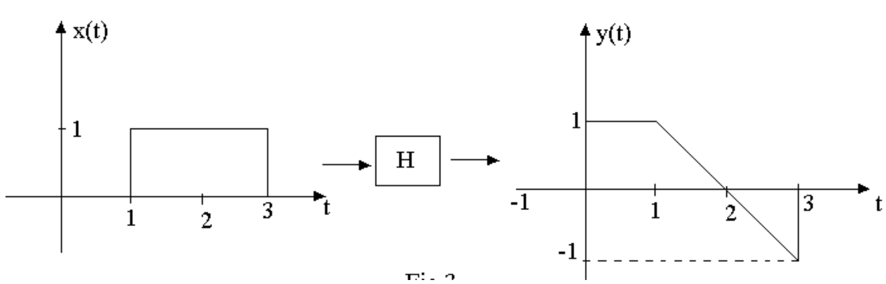



Odd part $x_o(t) = \dfrac{x(t) - x(-t)}{2}$




**Q.22.** A linear system H has an input-output pair as shown in Fig. Determine whether the system is causal and time-invariant. **(4)**

**Ans**



System is non-causal ∵ the output y(t) exists at t = 0 when input x(t) starts only at t = +1.
System is time-varying ∵ the expression for y(t) = [ u (t) – u (t-1)(t –1) + u (t –3) (t – 3) – u (t-3) ] shows that the system H has time varying parameters.

**Q.23**. Determine whether the system characterized by the differential equation $\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} + 2y(t) = x(t)$ is stable or not. **(4)**

**Ans**:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} + 2y(t) = x(t)$$

$y(t) \overset{L}{\longleftrightarrow} Y(s)$; $x(t) \overset{L}{\longleftrightarrow} X(s)$; Zero initial conditions

$s^2\, Y(s) - sY(s) + 2Y(s) = X(s)$

System transfer function $\frac{Y(s)}{X(s)} = \frac{1}{s^2 - s + 2}$ whose poles are in the right half plane.

Hence the system is not stable.

**Q.24** Determine whether the system $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ is invertible. **(5)**

**Ans:**

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Condition for invertibility: $H^{-1} H = I$ (Identity operator)

$$\begin{cases} H \longrightarrow \text{Integration} \\ H^{-1} \longrightarrow \text{Differentiation} \end{cases}$$

$x(t) \longrightarrow y(t) = H\{x(t)\}$

$H^{-1}\{y(t)\} = H^{-1}\, H\{x(t)\} = x(t)$

∴ The system is invertible.

**Q.25** Find the impulse response of a system characterized by the differential equation $y'(t) + a\, y(t) = x(t)$. **(5)**

**Ans:**

$y'(t) + a\, y(t) = x(t)$

$x(t) \overset{L}{\longleftrightarrow} X(s)$, $y(t) \overset{L}{\longleftrightarrow} Y(s)$, $h(t) \overset{L}{\longleftrightarrow} H(s)$

$sY(s) + aY(s) = X(s)$, assuming zero initial conditions

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s + a}$$

$\therefore$ The impulse response of the system is $h(t) = e^{-at} u(t)$

**Q.26.** Compute the Laplace transform of the signal $y(t) = (1 + 0.5 \sin t) \sin 1000t$. **(4)**

**Ans:**

$$y(t) = (1 + 0.5 \sin t) \sin 1000t$$
$$= \sin 1000t + 0.5 \sin t \sin 1000t$$
$$= \sin 1000t + 0.5 \left( \frac{\cos 999t - \cos 1001t}{2} \right)$$
$$= \sin 1000t + 0.25 \cos 999t - 0.25 \cos 1001t$$

$$\therefore Y(s) = \frac{1000}{s^2 + 1000^2} + 0.25 \frac{s}{s^2 + 999^2} - 0.25 \frac{s}{s^2 + 1001^2}$$

**Q.27.** Determine Fourier Transform $F(\omega)$ of the signal $f(t) = e^{-\alpha t} \cos(\omega t + \theta)$ and determine the value of $|F(\omega)|$. **(7)**

**Ans:**

We assume $f(t) = e^{-\alpha t} \cos(\omega t + \theta)\, u(t)$ because otherwise FT does not exist

$$f(t) \overset{FT}{\longleftrightarrow} F(\omega) = \int^{+\infty} e^{-\alpha t} \frac{e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}}{2} e^{-j\omega t} \, dt$$

$$\therefore F(\omega) = \frac{1}{2} \int^{+\infty} [e^{-\alpha t} e^{-j\omega t} e^{j\omega t + j\theta} + e^{-\alpha t} e^{-j\omega t} e^{-j\omega t - j\theta}] \, dt$$

$$= \frac{1}{2} \int^{+\infty} [e^{-\alpha t + j\theta} + e^{-j\theta} e^{-(\alpha + 2j\omega)t}] \, dt$$

$$|F(\omega)| = \frac{1}{2} \left| e^{j\theta} \frac{e^{-\alpha t}}{-\alpha} \Big|_0^{+\infty} + e^{-j\theta} \frac{e^{-(\alpha + 2j\omega)t}}{-(\alpha + 2j\omega)} \right|_0^{\omega}$$

$$= \frac{1}{2} \left| \frac{1}{\alpha} e^{j\theta} + \frac{1}{\alpha + 2j\omega} e^{-j\theta} \right|$$

$$\therefore |F(\omega)| = \frac{1}{2} \left| \frac{(\alpha + 2j\omega) e^{j\theta} + \alpha e^{-j\theta}}{\alpha (\alpha + 2j\omega)} \right|$$

$$= \frac{1}{2} \left| \frac{2\alpha \cos\theta + 2j\omega e^{j\theta}}{\alpha (\alpha + 2j\omega)} \right|$$

$$= \left| \frac{\alpha \cos\theta + j\, \omega \cos\theta - j\, \omega \sin\theta}{\alpha (\alpha + 2j\omega)} \right|$$

$$|F(\omega)|^2 = \frac{\alpha^2 \cos^2\theta + \omega^2 - 2\alpha\omega \sin\theta + \cos\theta}{\alpha^2 (\alpha^2 + 4\omega^2)}$$

$$= \frac{\omega^2 + \alpha^2 \cos^2\theta - \alpha\omega \sin 2\theta}{\alpha^2 (\alpha^2 + 4\omega^2)}$$

**Q.28.** Determine the impulse response h(t) and sketch the magnitude and phase response of the system described by the transfer function **(14)**

$$H(s) = \frac{s^2 + \omega_o^2}{s^2 + \frac{\omega_o}{Q} s + \omega_o^2}.$$

**Ans:**

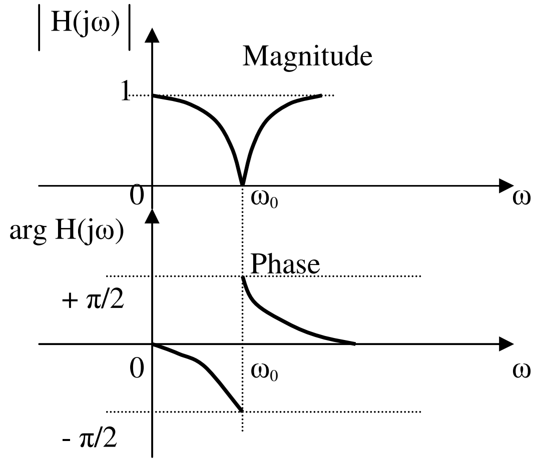
$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

$$H(j\omega) = \frac{(j\omega)^2 + \omega_0^2}{(j\omega)^2 + \frac{\omega_0}{Q}(j\omega) + \omega_0^2} = \frac{\omega_0^2 - \omega^2}{\omega_0^2 - \omega^2 + j\,\omega\,\frac{\omega_0}{Q}}$$

$$\therefore\ |H(j\omega)| = \frac{|\omega_0^2 - \omega^2|}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\left(\frac{\omega_0}{Q}\right)^2\right]^{1/2}}$$

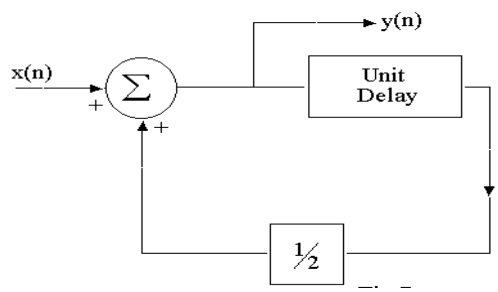
$$\text{Arg } H(j\omega) = -\tan^{-1}\left(\frac{\omega\left(\frac{\omega_0}{Q}\right)}{\omega_0^2 - \omega^2}\right)$$

| $\omega$ | $\lvert H(j\omega)\rvert$ | Arg $H(j\omega)$ |
|---|---|---|
| 0 | 1 | 0 |
| $\infty$ | 1 | 0 |
| $\omega_{0-}$ | 0 | $-\pi/2$ |
| $\omega_{0+}$ | 0 | $+\pi/2$ |



**Q.29**. Using the convolution sum, determine the output of the digital system shown in Fig. below.

Assume that the input sequence is $\{x(n)\} = \{\underset{\substack{\uparrow \\ n=0}}{3}, -1, 3\}$ and that the system is initially at rest. **(5)**




**Ans:**

$x(n) = \{\underset{\substack{\uparrow \\ n=0}}{3}, -1, 3\}$, system at rest initially (zero initial conditions)

$x(n) = 3\delta(n) - \delta(n-1) + 3\delta(n-2)$

$X(z) = 3 - z^{-1} + 3z^{-2}$

Digital system: $y(n) = x(n) + \frac{1}{2} y(n-1)$

$$\therefore Y(z) = \frac{X(z)}{1 - \frac{1}{2} z^{-1}} = \frac{3 - z^{-1} + 3z^{-2}}{1 - \frac{1}{2} z^{-1}} = -10 - 6 z^{-1} + \frac{13}{1 - \frac{1}{2} z^{-1}}$$

by partial fraction expansion.

Hence $y(n) = -10\,\delta(n) - 6\,\delta(n-1) + 13\left(\frac{1}{2}\right)^n u(n)$

**Q.30.** Find the z-transform of the digital signal obtained by sampling the analog signal $e^{-4t} \sin 4t\, u(t)$ at intervals of 0.1 sec. **(6)**

**Ans:**

$x(t) = e^{-4t} \sin 4t\, u(t), \qquad T = 0.1$ s

$x(n) = x(t \rightarrow nT) = x(0.1n) = (e^{-0.4})^n \sin(0.4n)$

$x(n) \overset{z}{\longleftrightarrow} X(z)$

$x(n) = \sin \Omega n\, u(n) \overset{z}{\longleftrightarrow} \dfrac{z \sin \Omega}{z^2 - 2z \cos \Omega + 1}$

$\alpha = e^{-0.4} = 0.6703, \ \frac{1}{\alpha} = 1.4918$

$\Omega = 0.4$ rad $= 22.92^\circ$

$\sin \Omega = 0.3894; \ \cos \Omega = 0.9211$

$$\alpha^n x(n) \overset{z}{\longleftrightarrow} X(z/\alpha)$$

$$\therefore\ X(z) = \frac{1.4918z\,(0.3894)}{(1.4918)^2 z^2 - 2(1.4918)z(0.9211) + 1}$$

$$X(z) = \frac{0.5809z}{2.2255\,z^2 - 2.7482z + 1}$$

**Q.31.** An LTI system is given by the difference equation $y(n) + 2y(n-1) + y(n-2) = x(n)$.

i. Determine the unit impulse response.

ii. Determine the response of the system to the input $(\underset{\substack{\uparrow \\ n=0}}{3}, -1, 3)$. **(4)**

**Ans:**

$y(n) + 2y(n-1) + y(n-2) = x(n)$

$Y(z) + 2z^{-1}\,Y(z) + z^{-2}\,Y(z) = X(z)$

$(1 + 2z^{-1} + z^{-2})Y(z) = X(z)$

(i). $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1 + 2z^{-1} + z^{-2}} = \dfrac{1}{(1 + z^{-1})^2}$ ( Binomial expansion)

$= 1 - 2z^{-1} + 3\,z^{-2} - 4\,z^{-3} + 5\,z^{-4} - 6\,z^{-5} + 7\,z^{-6} - \ldots\ldots$.( Binomial expansion)

$\therefore\ h(n) = \delta(n) - 2\delta(n-1) + 3\delta(n-2) - \ldots..$

$= \{\underset{\substack{\uparrow \\ n=0}}{1}, -2, 3, -4, 5, -6, 7, \ldots.\}$ is the impulse response.

(ii). $x(n) = \{\underset{\substack{\uparrow \\ n=0}}{3}, -1, 3\}$

$= 3\delta(n) - \delta(n-1) + 3\delta(n-2)$

$X(z) = 3 - z^{-1} + 3z^{-2}$

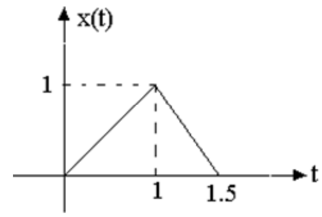
$\therefore\ Y(z) = X(z).H(z) = \dfrac{3 - z^{-1} + 3z^{-2}}{1 + 2z^{-1} + z^{-2}} = \dfrac{3(1 + 2z^{-1} + z^{-2}) - 7z^{-1}}{1 + 2z^{-1} + z^{-2}}$

$= 3 - 7\dfrac{z^{-1}}{(1 + z^{-1})^2}$

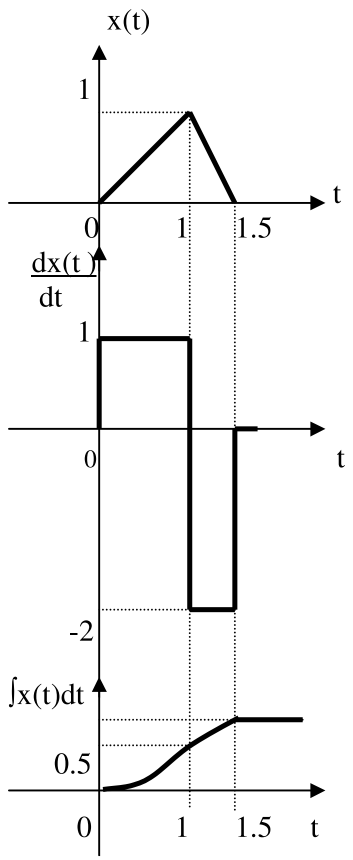
$\therefore\ y(n) = 3\delta(n) + 7nu(n)$ is the required response of the system.

**Q.32**. The signal x(t) shown below in Fig. is applied to the input of an
(i) ideal differentiator. (ii) ideal integrator.
Sketch the responses. **(1+4=5)**

$x(t) = t\,u(t) - 3t\,u(t-1) + 2t\,u(t-1.5)$




**Ans:**




(i) $0 < t < 1$

$$y(t) = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = 0.5 \text{ (Nonlinear)}$$

(ii) $1 < t < 1.5$

$$y(t) = y(1) + \int_1^t (3-2t)dt$$

$$= 0.5 + (3t - t^2)\Big|_1^t = 0.5 + 3t - t^2 - 3 + 1$$

$$= 3t - t^2 - 1.5 \quad \text{(Nonlinear)}$$

For t =1: $y(1) = 3 - 1 - 1.5 = 0.5$

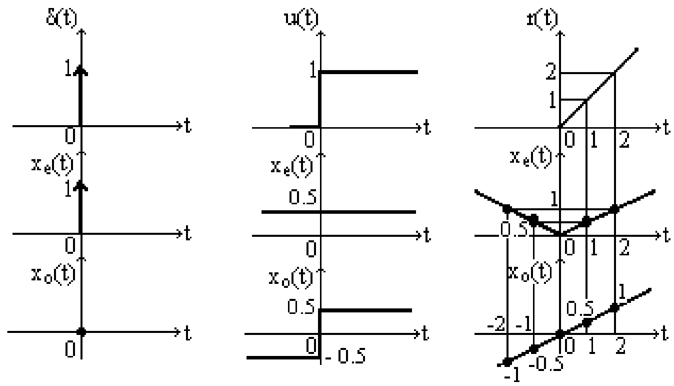
(iii) $t \geq 1.5$ : $y(1.5) = 4.5 - 2.25 - 1.5 = 0.75$

**Q.33.** Sketch the even and odd parts of **(1+2+3=6)**
(i) a unit impulse function (ii) a unit step function
(iii) a unit ramp function.

**Ans:**

Even part $x_e(t) = \dfrac{x(t) + x(-t)}{2}$

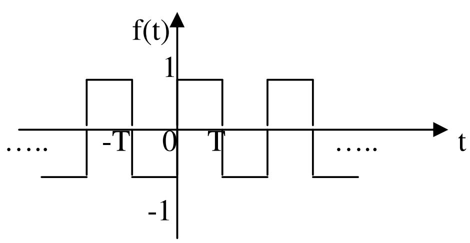
Odd part $x_o(t) = \dfrac{x(t) - x(-t)}{2}$




(i) unit impulse function (ii) unit step function (iii) unit ramp function

**Q.34.** Sketch the function $f(t) = u\left(\sin\frac{\pi t}{T}\right) - u\left(-\sin\frac{\pi t}{T}\right)$. (3)

**Ans:**




f(t) = { 1  0 < t T, 2T < t 3 T1
-1  T< t 2T, …
3  T < t <4T, ……

**Q.35.** Under what conditions, will the system characterized by $y(n) = \sum_{k=n_o}^{\infty} e^{-ak}\, x(n-k)$ be linear, time-invariant, causal, stable and memory less? (5)

**Ans:**

y(n) is : linear and time invariant for all k
causal if $n_0$ not less than 0.
stable if a > 0
memoryless if k = 0 only

**Q.36.** Let E denote the energy of the signal x (t). What is the energy of the signal x (2t)? (2)

**Ans:**

Given that

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

To find $E^1 = \int_{-\infty}^{\infty} |x(2t)|^2\, dt$

Let 2t =r then $E^1 = \int_{-\infty}^{\infty} |x(r)|^2 \frac{dr}{2} = \frac{1}{2}\int_{-\infty}^{\infty} |x(r)|^2\, dr = \frac{E}{2}$

**Q.37.** x(n), h(n) and y(n) are, respectively, the input signal, unit impulse response and output signal of a linear, time-invariant, causal system and it is given that $y(n-2) = x(n-n_1) * h(n-n_2)$, where * denotes convolution. Find the possible sets of values of $n_1$ and $n_2$. **(3)**

**Ans:**

$$y(n-2) = x(n-n_1) * h(n-n_2)$$

$$\therefore z^{-2}\,Y(z) = z^{-n1}\,X(z)\,.\,z^{-n2}\,H(z)$$

$$z^{-2}\,H(z)\,X(z) = z^{-(n_1+n_2)}X(z)H(z)$$

$$\therefore n_1+n_2 = 2$$

Also, $n_1, n_2 \geq 0$, as the system is causal. So, the possible sets of values for $n_1$ and $n_2$ are:

$\{n_1, n_2\} = \{(0,2),(1,1),(2,0)\}$

**Q.38.** Let h(n) be the impulse response of the LTI causal system described by the difference equation $y(n) = a\,y(n-1) + x(n)$ and let $h(n) * h_1(n) = \delta(n)$. Find $h_1(n)$. **(4)**

**Ans:**

$y(n) = a\,y(n-1) + x(n)$ and $h(n) * h_1(n) = \delta(n)$

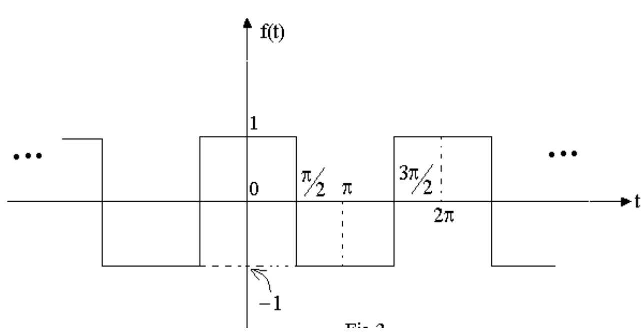
$Y(z) = az^{-1}\,Y(z) + X(z)$ and $H(z)\,H_1(z) = 1$

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1-az^{-1}}$ and $H_1(z) = \dfrac{1}{H(z)}$

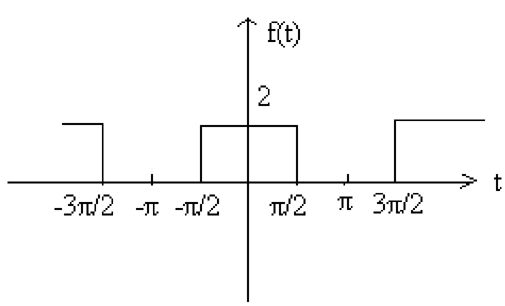
$\therefore H_1(z) = 1-az^{-1}$ or $h_1(n) = \delta(n) - a\,\delta(n-1)$

**Q.39.** Determine the Fourier series expansion of the waveform f (t) shown below in terms of sines and cosines. Sketch the magnitude and phase spectra. **(10+2+2=14)**

**Ans:**




Define g(t) = f(t) +1. Then the plot of g(t) is as shown , below and,




$\omega = 2\pi/2\pi = 1$

because $T = 2\pi$

$$g(t) = \begin{cases} 0 & -\pi < t < -\pi/2 \\ 2 & -\pi/2 < t < \pi/2 \\ 0 & \pi/2 < t < \pi \end{cases}$$

Let $g(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

Then $a_0 =$ average value of f(t) =1

$$a_n = \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} 2\cos nt\,dt = \frac{2}{\pi}\frac{\sin nt}{n}\Big|_{-\pi/2}^{\pi/2} = 2/n\pi \,.\, 2\sin n\pi/2$$

$= 4/n\pi \,.\, \sin n\pi/2$

$$= \begin{cases} 0 & \text{if } n = 2,4,6 \ldots\ldots \\ 4/n\pi & \text{if } n = 1,5,9 \ldots\ldots \\ -4/n\pi & \text{if } n = 3,7,11 \ldots\ldots \end{cases}$$

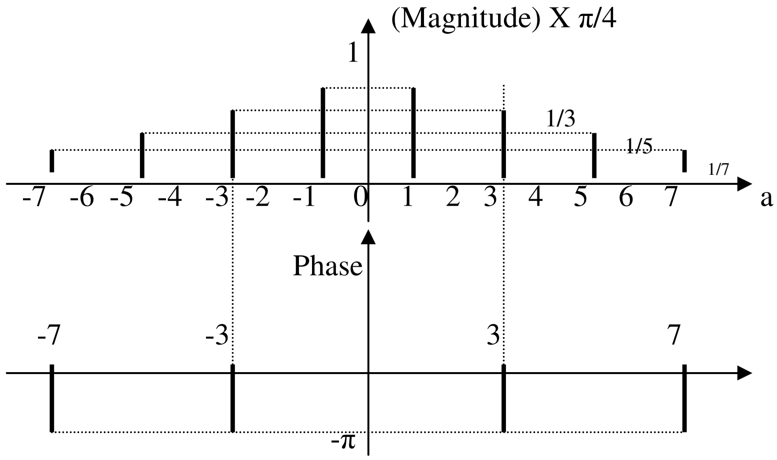
Also, $b_n = \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} 2\sin nt\,dt = \frac{4}{\pi}\frac{\cos nt}{n}\Big|_{\pi/2}^{\pi/2} = 4/n\pi[\cos n\pi/2 - \cos n\pi/2] = 0$

Thus, we have f(t) = -1 + g(t)

$$= \frac{4\cos t}{\pi} - \frac{4\cos 3t}{3\pi} + \frac{4\cos 5t}{5\pi} - \ldots\ldots$$

$= 4/\pi \;\{\cos t - \cos 3t/3 + \cos 5t/5 \;\ldots\ldots\}$

spectra :



**Q.40.** Show that if the Fourier Transform (FT) of x (t) is $X(\omega)$, then **(3)**

$$FT\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega).$$

**Ans:**

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) \text{ or } X(\omega)$$

$$\text{i.e., } x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \frac{d}{dt}[x(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

$$\therefore \frac{d}{dt}[x(t)] \overset{FT}{\longleftrightarrow} j\omega X(j\omega)$$

**Q.41.** Show, by any method, that $FT\left[\frac{1}{2}\right] = \pi\,\delta(\omega)$. **(2)**

**Ans:**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \pi\,\delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2} \qquad \because X(j\omega) = \pi\,\delta(\omega)$$

$$\therefore \frac{1}{2} \overset{FT}{\longleftrightarrow} \pi\,\delta(\omega)$$

**Q.42** Find the unit impulse response, h(t), of the system characterized by the relationship :

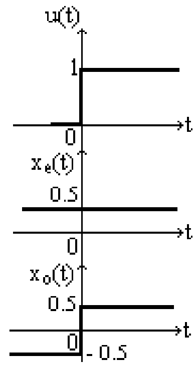
$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$ **(3)**

**Ans:**

$$y(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, & t \geq 0 = u(t) \\ 0, & \text{otherwise} \end{cases}$$

**Q.43.** Using the results of parts (a) and (b), or otherwise, determine the frequency response of the system of part (c). **(6)**

**Ans:**




As shown in the figure, u(t) = 1/2 + x(t)

where $x(t) = \begin{cases} 0.5, & t > 0 \\ -0.5, & t < 0 \end{cases}$

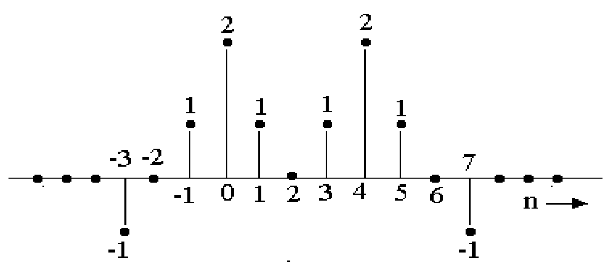
$\therefore$ dx/dt = δ (t) By (a) FT[δ (t)] = jωX(ω)

$\therefore$ X(ω) = 1/jω. Also FT[1/2] = πδ (ω)

Therefore FT [u(t)] = H(jω)= π√(ω) + 1/jω.

**Q.44.** Let $X\left(e^{j\omega}\right)$ denote the Fourier Transform of the signal x (n) shown below .**(2+2+3+5+2=14)**

**Ans:**




Without explicitly finding out $X\left(e^{j\omega}\right)$, find the following :-

(i) X (1)

(ii) $\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) d\omega$

(iii) X(-1)

(iv) the sequence y(n) whose Fourier Transform is the real part of $X\left(e^{j\omega}\right)$.

(v) $\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$.

**Ans:**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

+∞

(i) $X(1) = X(e^{j0}) = \sum_{-\infty} x(n) = -1 + 1 + 2 + 1 + 1 + 2 + 1 - 1 = 6$

(ii) $x(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$ ; $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega = 2\pi\, x(0) = 4\pi$

(iii) $X(-1) = X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x(n)\,(-1)^n = 1+0-1+2-1+0-1+2-1+0+1 = 2$

(iv) Real part $X(e^{j\omega}) \longleftrightarrow x_e(n) = \frac{x(n) + x(-n)}{2}$

$y(n) = x_e(n) = 0, \quad n < -7,\ n > 7$

$y(7) = \frac{1}{2}\, x(7) = -\frac{1}{2} = y(-7)$

$y(6) = \frac{1}{2}\, x(6) = 0 = y(-6)$

$y(5) = \frac{1}{2}\, x(5) = \frac{1}{2} = y(-5)$

$y(4) = \frac{1}{2}\, x(4) = 2 = y(-4)$

$y(3) = \frac{1}{2}[x(3) + x(-3)] = 0 = y(-3)$

$y(2) = \frac{1}{2}[x(2) + x(-2)] = 0 = y(-2)$

$y(1) = \frac{1}{2}[y(1) + y(-1)] = 1 = y(-1)$

$y(0) = \frac{1}{2}[\, y(0) + y(0)] = 2$

(v) Parseval's theorem:

$$\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} \left|x(n)\right|^2 = 2\pi(1 + 1 + 4 + 1 + 1 + 4 + 1 + 1) = 28\pi$$

**Q.45** If the z-transform of x (n) is X(z) with ROC denoted by $R_x$, find the z-transform of $y(n) = \sum_{k=-\infty}^{n} x(k)$ and its ROC. **(4)**

**Ans:**

$x(n) \overset{z}{\longleftrightarrow} X(z)$, RoC $R_x$

$y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=\infty}^{0} x(n-k) = \sum_{k=0}^{\infty} x(n-k)$

$\therefore Y(z) = X(z) \underbrace{\sum_{k=0}^{\infty} z^{-k}}_{\text{Geometric series}} = \frac{X(z)}{1 - z^{-1}}$, RoC at least $R_x \cap (|z| > 1)$

**Q.46** (i) x (n) is a real right-sided sequence having a z-transform X(z). X(z) has two poles, one of which is at $a\,e^{j\phi}$ and two zeros, one of which is at $r\,e^{-j\theta}$. It is also known that $\sum x(n) = 1$. Determine X(z) as a ratio of polynomials in $z^{-1}$. **(6)**

(ii) If $a = \frac{1}{2}, r = 2, \theta = \phi = \pi/4$ in part (b) (i), determine the magnitude of X(z) on the unit circle. **(4)**

**Ans:**

. (i) x(n) : real, right-sided sequence $\overset{z}{\longleftrightarrow}$ X(z)

$$X(z) = K\,\frac{(z - re^{-j\theta})(z - re^{j\theta})}{(z - ae^{j\Phi})(z - ae^{-j\Phi})} \quad ; \sum x(n) = X(1) = 1$$

$$= K\,\frac{z^2 - zr\,(e^{j\theta} + e^{-j\theta}) + r^2}{z^2 - za\,(e^{j\Phi} + e^{j\Phi}) + a^2}$$

$$= K\,\frac{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}{1 - 2a\cos\Phi z^{-1} + a^2 z^{-2}} = K.\frac{N(z^{-1})}{D(z^{-1})}$$

where $K.\dfrac{1 - 2r\cos\theta + r^2}{1 - 2a\cos\Phi + a^2} = X(1) = 1$

i.e., $K = \dfrac{1 - 2a\cos\Phi + a^2}{1 - 2r\cos\theta + r^2}$

(ii) $a = ½, r = 2, \theta = \Phi = \pi/4$ ; $K = \dfrac{1 - 2(½).(1/\sqrt{2}) + ¼}{1 - 2(2)(1/\sqrt{2}) + 4} = 0.25$

$$X(z) = (0.25)\,.\,\frac{1 - 2(2)(1/\sqrt{2})\,z^{-1} + 4z^{-2}}{1 - 2(½).(1/\sqrt{2})\,z^{-1} + ¼\,z^{-2}}$$

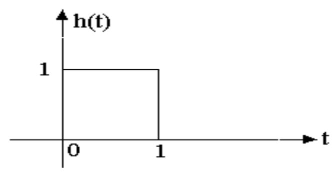
$$= (0.25)\,\frac{1 - 2\sqrt{2}\,z^{-1} + 4z^{-2}}{1 - (1/\sqrt{2})\,z^{-1} + ¼\,z^{-2}} \Longrightarrow X(e^{j\omega}) = (0.25)\,\frac{1 - 2\sqrt{2}\,e^{-j\omega} + 4\,e^{-2j\omega}}{1 - (1/\sqrt{2})\,e^{-j\omega} + ¼\,e^{-2j\omega}}$$

$$= \frac{-2\sqrt{2} + e^{j\omega} + 4\,e^{-j\omega}}{-2\sqrt{2} + 4e^{j\omega} + e^{-j\omega}}$$

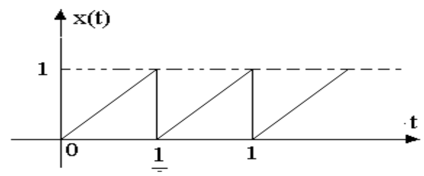
$$\therefore \left|X(e^{j\omega})\right| = 1$$

**Q.47** Determine, by any method, the output y(t) of an LTI system whose impulse response h(t) is of the form shown in fig(a). to the periodic excitation x(t) as shown in fig(b). **(14)**

**Ans:**




**Fig(a)**




**Fig(b)**

$h(t) = u(t) - u(t-1) \Rightarrow H(s) = \frac{1-e^{-s}}{s}$

First period of x(t), $x_T(t) = 2t\,[u(t) - u(t-½)]$

$= 2[\,t\,u(t) - (t-1/2)\,u(t-1/2) - 1/2\,u(t-1/2)]$

$\therefore X_T(s) = 2[1/s^2 - e^{-s/2}/s^2 - 1/2\,e^{-s/2}/s]$

$X(s) = X_T(s) / 1 - e^{-s/2}$

$$Y(s) = \frac{1-e^{-s}}{s}\cdot\frac{1}{1-e^{-s/2}}\,2\left(\frac{1-e^{-s/2}-0.5se^{-s/2}}{s^2}\right)$$

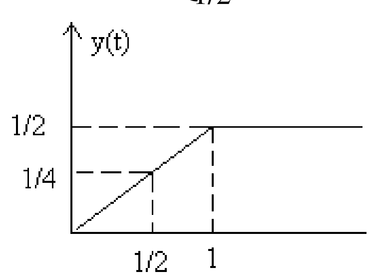
$$= \frac{2}{s^3}\left(1+e^{-s/2}\right)\left[1-e^{-s/2}-0.5\,s\,e^{-s/2}\right]$$

$$= \frac{2}{s^3}\left(1-e^{-s}-0.5s(e^{-s/2}+e^{-s})\right)$$

$$= 2\,\frac{1-e^{-s}}{s^3} - \frac{e^{-s/2}+e^{-s}}{s^2}$$

Therefore $y(t) = t^2\,u(t) - (t-1)^2\,u(t-1) - \left(t-\frac{1}{2}\right)u\left(t+\frac{1}{2}\right) - (t-1)u(t-1)$

This gives
$$y(t) = \begin{cases} t2 & 0 < t < 1/2 \\ t^2 - t + 1/2 & 1/2 < t < 1 \\ 1/2 & t > 1 \end{cases}$$




(not to scale)

**Q.48** Obtain the time function f(t) whose Laplace Transform is $F(s)=\dfrac{s^2+3s+1}{(s+1)^3(s+2)^2}$. **(14)**

**Ans:**

$$F(s) = \frac{s^2+3s+1}{(s+1)^3(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{D}{(s+2)} + \frac{E}{(s+2)^2}$$

$$A(s+2)^2(s+1)^2 + B(s+2)^2(s+1) + C(s+2)^2 + D(s+1)^3(s+2) + E(s+1)^3 = s^2+3s+1$$

$$C = \left.\frac{s^2+3s+1}{(s+2)^2}\right|_{s=-1} = \frac{1-3+1}{1} = -1 \qquad \boxed{C=-1}$$

$$E = \left.\frac{s^2+3s+1}{(s+1)^3}\right|_{s=-2} = \frac{4-6+1}{-1} = 1 \qquad \boxed{E=1}$$

$$A(s^2+3s+2)^2 + B(s^2+4s+4)(s+1) + C(s^2+4s+4) + D(s^3+3s^2+3s+1)(s+2) + E(s^3+3s^2+3s+1) = s^2+3s+1$$

$$A(s^4+6s^3+13s^2+12s+4) + B(s^3+5s^2+8s+4) + C(s^2+4s+4) + D(s^4+5s^3+9s^2+7s+2) + E(s^3+3s^2+3s+1) = s^2+3s+1$$

$s^4$ : $A+D = 0$

$s^3$ : $6A+B+5D+E = 0$ ; $A+B+1 = 0$ as $5(A+D) = 0, E = 1$

$s^2$ : $13A+5B+C+9D+3E = 1$ ; $4A+5B+1 = 0$ as $9(A+D) = 0, C = -1, E = 1$

$s^1$ : $12A+8B+4C+7D+3E = 3$ ; $5A+8B-4 = 0$ as $7(A+D) = 0, C = -1, E = 1$

$s^0$ : $4A+4B+4C+2D+E = 1$

$A+B = -1$ ; $4(A+B)+B+1 = 0$ or $-4+B+1 = 0$ or $\boxed{B = 3}$

$\boxed{A = -4}$

$\therefore A = -1-3 = -4$

$A+D = 0$ or $D = -A = 4$ $\boxed{D = 4}$

$$\therefore F(s) = \frac{-4}{(s+1)} + \frac{3}{(s+1)^2} + \frac{-1}{(s+1)^3} + \frac{4}{(s+2)} + \frac{1}{(s+2)^2}$$

$$\therefore f(t) = L^{-1}[F(s)] = -4e^{-t} + 3t\,e^{-t} - t^2 e^{-t} + 4e^{-2t} + t\,e^{-2t} = [e^{-t}(-4+3t-t^2) + e^{-2t}(4+t)]\,u(t)$$

$$\therefore f(t) = [e^{-t}(-4+3t-t^2) + e^{-2t}(4+t)]\,u(t)$$

**Q.49** Define the terms variance, co-variance and correlation coefficient as applied to random variables. **(6)**

**Ans:**

**Variance** of a random variable X is defined as the second central moment $E[(X-\mu_X)]^n$, n=2, where central moment is the moment of the difference between a random variable X and its mean $\mu_X$ i.e.,

$$\sigma_X^2 = \text{var}\,[X] = \int_{-\infty}^{+\infty} (x-\mu_X)^2 f_x(x)\,dx$$

**Co-variance** of random variables X and Y is defined as the joint moment:

$$\sigma_{XY} = \text{cov}\,[XY] = E[\{X-E[X]\}\{Y-E[Y]\}] = E[XY]-\mu_X\mu_Y$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

**Correlation coefficient** $\rho_{XY}$ of X and Y is defined as the co-variance of X and Y normalized w.r.t $\sigma_X\sigma_Y$ :

$$\rho_{XY} = \frac{\text{cov}\,[XY]}{\sigma_X\sigma_Y} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$$