TYPICAL QUESTIONS & ANSWERS

PART – I

OBJECTIVE TYPE QUESTIONS

Each Question carries 2 marks:

Choose the correct or best alternative in the following:

Q.1 The points 2i – j + k, i – 3j – 5k, 3i – 4j – 4k are the vertices of a triangle which is

(A) equilateral. **(B)** isosceles.
(C) right angled. **(D)** None of these.

Ans: C

$OA = 2\hat{i} - \hat{j} + \hat{k} \quad , \quad O\vec{B} = \hat{i} - 3\hat{j} - 5\hat{k} \quad , \quad O\vec{C} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

$A\vec{B} = O\vec{B} - O\vec{A} = -\hat{i} - 2\hat{j} - 6\hat{k} \Rightarrow \left|A\vec{B}\right| = \sqrt{41}$

$B\vec{C} = O\vec{C} - O\vec{B} = 2i - \hat{j} + \hat{k} \Rightarrow \left|B\vec{C}\right| = \sqrt{6}$

$A\vec{C} = O\vec{C} - O\vec{A} = \hat{i} - 3\hat{j} - 5\hat{k} \Rightarrow \left|A\vec{C}\right| = \sqrt{35}$

$\therefore (AB)^2 = BC^2 + AC^2$

Thus Δ is right angled

Q.2 If $\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)^{48} = 3^{24}(x + iy)$, then ordered pair (x, y) is

(A) (0, 2). **(B)** (0, 1).
(C) (1, 0). **(D)** (1, 1).

Ans: C

$$\left\{\frac{3}{2} + i\frac{\sqrt{3}}{2}\right\}^{48} = 3^{24}(x + iy)$$

$$\Rightarrow \left(\sqrt{3}\right)^{48}\left\{\frac{\sqrt{3}}{2} + i\frac{1}{2}\right\}^{48} = 3^{24}(x + iy)$$

$$\Rightarrow \left\{\frac{\sqrt{3}}{2} + i\frac{1}{2}\right\}^{48} = x + iy$$

$$\Rightarrow \left\{\cos\frac{\pi}{3}\times 48 + i\sin\frac{\pi}{3}\times 48\right\} = x + iy$$

$$\Rightarrow \{1 + i\times 0\} = x + iy$$

i.e. Pair (x, y) is (1, 0).

Q.3 If $2\cos\theta = x + \frac{1}{x}$, $2\cos\phi = y + \frac{1}{y}$ then $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ is

(A) $2\sin(m\theta + n\phi)$. **(B)** $2\sin(m\theta - n\phi)$.
(C) $2\cos(m\theta + n\phi)$. **(D)** $2\cos(m\theta - n\phi)$.

Ans: D

$$\frac{X^m}{Y^n} = (\cos m\theta + i\sin m\theta).(\cos n\phi + i\sin n\phi)^{-1}$$

$$= (\cos m\theta + i\sin m\theta).(\cos(-n\phi) + i\sin(-n\phi))$$

$$\frac{X^m}{Y^n} = \cos(m\theta - n\phi) + i\sin(m\theta - n\phi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Similarly $\frac{Y^n}{X^m} = \cos(m\theta - n\phi) - i\sin(m\theta - n\phi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$

Adding equation (1) and (2) we get

$$\frac{X^m}{Y^n} + \frac{Y^n}{X^n} = 2\cos(m\theta - n\phi)$$

Q.4 A vector of magnitude 2 along a bisector of the angle between the two vectors 2i - 2j + k and i + 2j - 2k is

(A) $\frac{2}{\sqrt{10}}(3i - k)$. **(B)** $\frac{1}{\sqrt{26}}(i - 4j + 3k)$.
(C) $\frac{2}{\sqrt{26}}(i - 4j + 3k)$. **(D)** None of these.

Ans: A

Let $\bar{a}_0$ and $\bar{b}_0$ be unit vectors along a and b respectively. $\bar{a}_0 = \frac{1}{3}(2i - 2j + k)$, $\bar{b}_0 = \frac{1}{3}(i + 2j - 2k)$

Required vector $\bar{c} = \frac{\lambda}{3}(3i - k)$. $4 = \lambda^2.\frac{10}{9}$

$$\lambda = \frac{6}{\sqrt{10}}$$

Thus $\bar{c} = \frac{2}{\sqrt{10}}(3i - k)$

Q.5 Let A and B be two matrices such that $A \neq 0$ and AB =0. Then we must have

(A) $B = 0$. **(B)** B to be identity matrix.
(C) $B = -A$. **(D)** None of these.

Ans: D

Q.6 If $f(x)=\begin{vmatrix} \frac{1}{\sqrt{2}} & \sin x & 1 \\ \frac{1}{\sqrt{2}} & \cos x & x \\ 1 & 1 & x^2 \end{vmatrix}$, then $f\left(\frac{\pi}{4}\right)$ is

(A) 0. **(B)** 1.
(C) 2. **(D)** 3.

Ans: A

$$f(x)=\begin{vmatrix} \frac{1}{\sqrt{2}} & \text{Sin}x & 1 \\ \frac{1}{\sqrt{2}} & \text{Cos}x & x \\ 1 & 1 & x^2 \end{vmatrix}$$

$$f\left(\frac{\pi}{4}\right)=\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\pi}{4} \\ 1 & 1 & \frac{\pi^2}{4^2} \end{vmatrix}$$

Since c_1 & c_2 are same $\therefore$ $f\left(\frac{\pi}{4}\right)=0$

Q.7 $L^{-1}\left(\frac{1}{s^n}\right)$ exists only when n is

(A) zero. **(B)** –ve integer.
(C) +ve integer. **(D)** –ve rational.

Ans: C

$L^{-1}\left(\frac{1}{s^n}\right)=\frac{t^{n-1}}{\lfloor n-1}$, n is positive integer.

Q.8 The differential equation of the curve $y=a\cos(x-b)$, where a and b are constants, is

(A) $\frac{d^2y}{dx^2}-y=0$. **(B)** $\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=0$.
(C) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$. **(D)** $\frac{d^2y}{dx^2}+y=0$.

Ans: D

Since y = a cos(x – b)

$\therefore \frac{dy}{dx} = -a\sin(x-b)$, $\frac{d^2y}{dx^2} = -a\cos(x-b) = -ay$

$\Rightarrow \frac{d^2y}{dx^2} + y = 0$

Q.9 If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are vectors then $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)$ is equal to

(A) $\vec{a}\cdot\vec{c} - \vec{b}\ \vec{d}$

(B) $\vec{a}\times\vec{c} - \vec{b}\times\vec{d}$

(C) $\left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right) - \left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$

(D) none of above.

Ans: C

$\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$

$= \left(\vec{a}.\vec{c}\right)\left(\vec{b}.\vec{d}\right) - \left(\vec{b}.\vec{c}\right)\left(\vec{a}.\vec{d}\right)$

Q.10 If A, B are square matrices of the same size then

(A) $(AB)^t = A^tB^t$

(B) $(AB)^t = B^tA^t$

(C) $(AB)^t = A\,B$

(D) $(AB)^t = B\,A$

Ans: B

By definition

$(AB)^t = B^t.A^t$

Q.11 If z_1 and z_2 are two complex numbers then $|z_1 + z_2|$ is

(A) $= |z_1| + |z_2|$

(B) $\le |z_1| + |z_2|$

(C) $\le |z_1| - |z_2|$

(D) $\ge |z_1| + |z_2|$

Ans: B

$\because |Z_1 + Z_2| \le |Z_1| + |Z_2|$

(Triangle inequality)

Q.12 The value of $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}$ is equal to

(A) $3a^2x$ **(B)** $a^2(3x - a)$
(C) $a^2(3x + a)$ **(D)** $3ax^2$

Ans: C

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3 = \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix}$$

$$= (3x+a) = \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix}$$

$C_2 \to C_2 - C_1, \quad C_3 \to C_3 - C_1$

$$= (3x+a) = \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = a^2(3x+a)$$

Q.13 If $I+A+A^2+\ldots+A^K=0$, then A^{-1} is equal to

(A) A^K **(B)** A^{K-1}
(C) A^{K+1} **(D)** $I+A$

Ans: A

If $1 + A + A^2 + ...+ A^k = 0$ (Characteristic equation of Matrix)

$\Rightarrow A^{-1} + I + A + A^2 + ...+ A^{(k-1)} = 0$ (Divided by A)

$\Rightarrow A^{-1} + I + A + A^2 + ...+ A^{(k-1)} + A^k = A^k$

$\Rightarrow A^{-1} + 0 = A^k$

$A^{-1} = A^k$

Q.14 If A is any real square matrix then $A+A^t$ is

(A) Hermitian. **(B)** Skew-hermition.
(C) Symmetric. **(D)** Skew-symmetric.

Ans: C

$(A + A^t)^t = A^t + (A^t)^t = A^t + A$

Q.15 The Laplace transform $L(t^n)$ is

(A) $\frac{n!}{s^n}$. **(B)** $\frac{n!}{s^{n+1}}$.

(C) $\frac{1}{s}$. **(D)** $\frac{s^n}{n!}$

Ans: B

$$L\{t^n\} = \int_0^\infty e^{-st}..t^n dt = \frac{!n}{s^{n+1}}$$

Q.16 The solution of differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ is

(A) $y = (c_1 + c_2x)e^x$ **(B)** $y = (c_1 + c_2x)e^{2x}$.

(C) $y = (c_1 + c_2x)e^{3x}$. **(D)** $(c_1 + c_2x)e^{-3x}$

Ans: C

A.E $m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0$ $m = 3,3$ Roots are real and equal.

$\therefore c.f = (C_1 + xC_2)e^{3x}$ and P.I = 0

$Y = (C_1 + C_2x)e^{3x}$

Q.17 The value of a_0 in the Fourier series $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + ...a_n \cos nx + ...$ is given by

(A) $\frac{1}{\pi}\int_0^{2\pi} f(x)dx$ **(B)** $\frac{1}{2\pi}\int_0^{2\pi} f(x)dx$

(C) $\frac{1}{\pi}\int_0^{\pi} f(x)dx$ **(D)** 0

Ans: A

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$ By definition

Q.18 The inverse Laplace transform $L^{-1}\left(\frac{4}{s-2}\right)$ is

(A) e^t **(B)** $2e^{2^t}$

(C) $4e^{2t}$ **(D)** $4e^{4t}$

Ans: C

$$L^{-1}\left(\frac{4}{\delta - 2}\right) = 4e^{2t}L^{-1}\left\{\frac{1}{s}\right\} = 4e^{2t}.1 = 4e^{2t}$$

Q.19 Let $z_1 = 2 - 5i$; $z_2 = -1 + 4i$; $z_3 = 6 + i$ and $z_4 = 3 - 7i$. Express $\frac{(z_1 + \overline{z_2})z_3}{z_4}$ in the form a + bi, a , b $\in$ R.

(A) $\frac{208}{29} + \frac{27}{29}i$ **(B)** $\frac{208}{29} - \frac{27}{29}i$

(C) $\frac{28}{209}+\frac{27}{29}i$ **(D)** $\frac{28}{209}-\frac{27}{29}i$

Ans: B

Q.20 The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1-i\sqrt{3}}{2}$ are vertices of the a triangle which is

(A) acute-angled and isosceles **(B)** right-angled and isosceles
(C) obtuse-angled and isosceles **(D)** equilateral

Ans: D

Q.21 A unit vector parallel to 3i+4j-5k is

(A) $-\frac{3}{5\sqrt{2}}i-\frac{4}{5\sqrt{2}}j+\frac{1}{\sqrt{2}}k$ **(B)** $\frac{3}{5\sqrt{2}}i-\frac{4}{5\sqrt{2}}j-\frac{2}{\sqrt{2}}k$
(C) $-\frac{3}{5\sqrt{2}}i+\frac{4}{5\sqrt{2}}j+\frac{2}{\sqrt{2}}k$ **(D)** $\frac{3}{5\sqrt{2}}i-\frac{4}{5\sqrt{2}}j+\frac{1}{\sqrt{2}}k$

Ans: A

Q.22 Let $\vec{a}$ = (1, 2, 0), $\vec{b}$ = (-3, 2, 0), $\vec{c}$ = (2, 3, 4). Then $\vec{a}\cdot(\vec{b}\times\vec{c})$ equals

(A) 33 **(B)** 30
(C) 31 **(D)** 32

Ans: D

Q.23 If ω is complex cube root of unity, and $A = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$, then A^{100} is equal to

(A) 0 **(B)** -A
(C) A **(D)** none of these

Ans: C

Q.24 If A and B are symmetric matrices, then AB + BA is a

(A) diagonal matrix **(B)** null matrix
(C) symmetric matrix **(D)** Skew-symmetric matrix

Ans: C

Q.25 The function $x^3 \sin x$ is

(A) odd **(B)** even
(C) neither **(D)** none of these

Ans: B

Q.26 The function cos x + sin x + tan x + cot x + sec x + cosecx is

(A) both periodic and odd **(B)** both periodic and even

(C) periodic but neither even nor odd **(D)** not periodic

Ans: C

Q.27 The Laplace Transform for $\sin at$ is

(A) $\frac{s}{s^2 - a^2}$ **(B)** $\frac{a}{s^2 + a^2}$

(C) $\frac{s}{s^2 + a^2}$ **(D)** $\frac{a}{s^2 - a^2}$

Ans: B

Q.28 The Inverse Laplace Transform for $\frac{s+9}{s^2+6s+13}$ is

(A) $e^{3t}(\cos(2t) + 3\sin(2t))$ **(B)** $e^{-3t}(\cos(2t) + 3\sin(2t))$

(C) $e^{3t}(\cos(2t) - 3\sin(2t))$ **(D)** $e^{-3t}(\cos(2t) - 3\sin(2t))$

Ans: A

Q.29 The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is

(A) 8 **(B)** 12
(C) 16 **(D)** None of these

Ans: D

Q.30 A square root of 3 + 4i is

(A) $\sqrt{3} + i$ **(B)** $2 - i$
(C) $2 + i$ **(D)** None of these

Ans: C

Q.31 Any vector a is equal to

(A) $(a \cdot \hat{i})\hat{i} + (a \cdot \hat{j})\hat{j} + (a \cdot \hat{k})\hat{k}$ **(B)** $(a \cdot \hat{j})\hat{i} + (a \cdot \hat{k})\hat{j} + (a \cdot \hat{i})\hat{k}$

(C) $(a \cdot \hat{k})\hat{i} + (a \cdot \hat{i})\hat{j} + (a \cdot \hat{j})\hat{k}$ **(D)** $(a \cdot a)(\hat{i} + \hat{j} + \hat{k})$

Ans: A

Q.32 If a and b are two unit vectors inclined at an angle θ and are such that a + b is a unit vector, then θ is equal to

(A) $\pi/4$ **(B)** $\pi/3$

(C) $\pi/2$ **(D)** $2\pi/3$

Ans: D

Q.33 The value of the determinant $\begin{vmatrix} 1 & \omega^3 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^4 & 1 \end{vmatrix}$, where ω is an imaginary cube root of unity is

(A) $(1-\omega)^2$ **(B)** 3
(C) -3 **(D)** 4

Ans: B

Q.34 The value of the determine $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ is equal to

(A) -4 **(B)** 0
(C) 1 **(D)** 4

Ans: D

Q.35 The inverse of a diagonal matrix is
(A) not defined **(B)** a skew-symmetric matrix
(C) a diagonal matrix **(D)** a unit matrix

Ans: C

Q.36 The period of function sin 2x + cot 3x + sec 5x is
(A) π **(B)** 2π
(C) $\pi/2$ **(D)** $\pi/3$
Ans: B

Q.37 The Laplace transform of $\sin^2 t$ is

(A) $\frac{2}{s(s^2+4)}$ **(B)** $\frac{1}{s(s^2+4)}$

(C) $\frac{2}{(s+4)(s-2)}$ **(D)** $\frac{1}{(s+4)(s-2)}$

Ans: A

Q.38 The solution of the differential equation $(D^2+4)y = e^x$ is

(A) $c_1 \cos 2x - c_2 \sin 2x + \frac{e^x}{4}$ **(B)** $c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{4}$

(C) $c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{5}$ **(D)** $c_1 \cos 4x - c_2 \sin 4x + \frac{e^x}{5}$

Ans: C

Q.39 Modules of $\left(\sqrt{i}\right)^{\sqrt{i}}$ is

(A) $e^{\pi/4}$ **(B)** $e^{-\pi/4}$

(B) $e^{-\pi/4\sqrt{2}}$ **(D)** $e^{\pi/4\sqrt{2}}$

Ans: A

Let $x+iy=\left(\sqrt{i}\right)^{\sqrt{i}}$

$\log(x+iy)=\sqrt{i}\log\sqrt{i}$

$\Rightarrow \log(x+iy)=\frac{1}{2}\sqrt{i}\log i$

$\Rightarrow \log(x+iy)=\frac{1}{2}\sqrt{i}\left\lfloor \log i + i\tan^{-1}\infty \right\rfloor$

$\Rightarrow \log(x+iy)=\frac{1}{2}\sqrt{i}(i\tan^{-1}\infty)$

$\Rightarrow \log(x+iy)=\frac{1}{2}\sqrt{i}\left(i\frac{\pi}{2}\right)$

$\Rightarrow \log(x+iy)=i^{3/2}\frac{\pi}{4}$

$\Rightarrow (x+iy)=e^{i^{3/2}\frac{\pi}{4}}$

$\therefore$ *Modulus of* $\left(\sqrt{i}\right)^{\sqrt{i}}=e^{\frac{\pi}{4}}$

Q.40 If $\tan\frac{x}{2}=\tanh\frac{y}{2}$ then the value of $\cos x \cosh y$ is

(A) –1 **(B)** 0

(C) 1/2 **(D)** 1

Ans: D

Q.41 The two non-zero vectors $\overline{A}$ and $\overline{B}$ are parallel if

(A) $\overline{A}\times\overline{B}=0$ (B) $\left|\overline{A}\times\overline{B}\right|=1$

(C) $\overline{A}\cdot\overline{B}=0$ **(D)** $\left|\overline{A}\right|=\left|\overline{B}\right|$

Ans: A

Two non-zero vector $\overline{A}$ and $\overline{B}$ are parallel if $\overline{A}\times\overline{B}=0$ $(\because \sin\theta=0)$

Q.42 The volume of the parallelopipid with sides $\overline{A}=6\hat{i}-2\hat{j}$, $\overline{B}=\hat{j}+2\hat{k}$, $\vec{C}=\hat{i}+\hat{j}+\hat{k}$ A is

(A) 5 cubic units **(B)** 10 cubic units

(C) 15 cubic units **(D)** 20 cubic units

Ans: B

Volume of parallelepiped with sides $\left(6\hat{i}-2\hat{j}\right),\left(\hat{j}+2k\right),\left(\hat{i}+j+k\right)$

$$\begin{vmatrix} 6 & -2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 10 \text{ cubic units}$$

Q.43 If $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ then eigen value of A^{-1} are

(A) $1, 1/2, 1/3$ **(B)** 1, 2, 3

(C) 0, 1, 2 **(D)** $0, 1, 1/2$

Ans: A

Let A = $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix}$

Eigen values of A are 1, 2, 3

$\therefore$ eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$

$\Rightarrow \lambda = 1, \frac{1}{2}, \frac{1}{3}$

Q.44 The sum and product of the eigen values of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are

(A) Sum = 5, Product = 7 **(B)** Sum = 7, Product = 5
(C) Sum = 5, Product = 5 **(D)** Sum = 7, Product = 7

Ans: B

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-5) = 0$$

$$\Rightarrow \lambda = 1,1,5$$

Sum of Eigen value = 07
Product of Eigen value = 5

Q.45 If $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ then the value of f(0) is

(A) 0 **(B)** $\pi/2$

(C) $-\pi/2$ **(D)** π

Ans: C

Zero is the point of discontinuously

$$f(0)=\frac{f(0-)\,|\,f(0+)}{2}$$

$$=\frac{-\pi+0}{2}=-\pi/2$$

$$f(0)=-\frac{\pi}{2}$$

Q.46 The inverse Laplace transform of $(s+2)^{-2}$

(A) e^{-2t} **(B)** e^{2t}

(C) te^{2t} **(D)** te^{-2t}

Ans: D

$$L^{-1}\left\{\frac{1}{(s+2)^2}\right\}=e^{-2t}\quad L^{-1}\left\{\frac{1}{s^2}\right\}$$ by first shifting theorem

$$=e^{-2t}t$$

Q.47 The solution of the differential equation $y''+y=0$ satisfying the condition $y(0)=1$, $y(\pi/2)=2$ is

(A) $y=2\cos x+\sin x$ **(B)** $y=\cos x+2\sin x$

(C) $y=\cos x+\sin x$ **(D)** $y=2(\cos x+\sin x)$

Ans: B

$$y''+y=0\Rightarrow(D^2+1)y=0$$

$$m=\pm i$$

c.f = $(c_1\cos x+c_2\sin x)$

putting x = 0, y(0) = 1

$$c_1=1$$

Putting $x=\frac{\pi}{2},\quad y\left(\frac{\pi}{2}\right)=2$

$$c_2=2$$

$$\therefore[y=\cos x+2\sin x]$$

Q.48 Fourier Sine transform of 1/x is

(A) S **(B)** S/2

(C) $S^2/2$ **(D)** $-S^2/2$

Ans: C

Q.49 The complex numbers Z = x + iy, which satisfy the equation $\left|\frac{Z-5i}{Z+5i}\right|=1$ lie on

(A) the x-axis.
(B) the line y = 5.
(C) A circle passing through the origin.
(D) None of these.

Ans: A

$$\left|\frac{x+i(y-5)}{x+i(y+5)}\right|=1$$
$$\Rightarrow \sqrt{x^2+(y-5)^2}=\sqrt{x^2+(y+5)^2}$$
$$\Rightarrow (y-5)^2=(y+5)^2$$
$\Rightarrow y=0$ i.e x-axis

Q.50 If $Z^2=|iZ|^2$, then

(A) $\text{Re}(Z)=0$
(B) $\text{Im}(Z)=0$
(C) Z=0
(D) $Z=x(1\pm i)$, with x real

Ans: B

Given $z^2=|iz|^2$
$$\Rightarrow (x+iy)^2=|i(x+iy)|^2$$
$$\Rightarrow (x+iy)^2=|ix-y|^2$$
$$\Rightarrow \cancel{x^2}-y^2+2ixy=\cancel{x^2}+y^2$$
$$\Rightarrow 2ixy-2y^2=0$$
$$\Rightarrow 2y(ix-y)=0 \quad \Rightarrow y=0$$
$$\Rightarrow \text{Im}(z)=0$$

Q.51 If $\vec{a}$ and $\vec{b}$ are two unit vectors and ϕ is the angle between them, then $\left(\frac{1}{2}\right)|\vec{a}-\vec{b}|$ is equal to

(A) $\pi/2$
(B) 0
(C) $|\text{Sin}\,\phi/2|$
(D) $|\text{Cos}\,\phi/2|$

Ans: C

Given $\overline{a},\overline{b}$ are Unit vector
$$|a|=1, |b|=1, \quad \Rightarrow a.b=\cos\phi$$
Now $|\overline{a}-\overline{b}|^2=a^2+b^2-2ab$
$$\Rightarrow |\overline{a}-\overline{b}|^2=1+1-2\cos\phi$$
$$=2(1-\cos\phi)$$

$$= 2\left(2\sin^2\frac{\phi}{2}\right)$$

$$= 4\sin^2\frac{\phi}{2}$$

$$\Rightarrow \left|\bar{a}-\bar{b}\right| = 2\left|\sin\frac{\phi}{2}\right|$$

$$\Rightarrow \frac{1}{2}\left|\bar{a}-\bar{b}\right| = \left|\sin\frac{\phi}{2}\right|$$

Q.52 A vector which makes equal angles with the vectors $(1/3)(\hat{i}-2\hat{j}+2\hat{k})$, $(1/5)(-4\hat{i}-3\hat{k})$ and $\hat{j}$ is

(A) $5\hat{i}+\hat{j}+5\hat{k}$ **(B)** $-5\hat{i}+\hat{j}+5\hat{k}$

(C) $-5\hat{i}-\hat{j}+5\hat{k}$ **(D)** $5\hat{i}+\hat{j}-5\hat{k}$

Ans: B

Let vector be $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\frac{\vec{a}.\frac{1}{3}(\hat{i}-2j+2\hat{k})}{|a|} = \cos\theta$$

$$\frac{\vec{a}\left(-\frac{4}{5}\hat{i}-\frac{3}{5}\hat{k}\right)}{|a|} = \cos\theta$$

$$\frac{\vec{a}\hat{j}}{|a|} = \cos\theta$$

$$\therefore \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}).\frac{1}{3}(\hat{i}-2\hat{j}+2\hat{k})}{|a|} = \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}).(-\frac{4}{5}\hat{i}-\frac{3}{5}\hat{k})}{|a|}$$

$$= \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}).\hat{j}}{|a|}$$

$$\Rightarrow \frac{a_1}{3}-\frac{2a_2}{3}+\frac{2a_3}{3} = -\frac{4}{5}a_1-\frac{3}{5}a_3 = a_2$$

Let $a_2 = t$ then $a_3 = 5t, a_1 = -5t$

$\therefore \vec{a} = -5\hat{i}+\hat{j}+5\hat{k}$

Q.53 If ω $(\neq 1)$ is a cube root of unity and $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$, then

(A) x = 1 **(B)** $x = \omega$

(C) $x = \omega^2$ **(D)** none of these

Ans: D

$$\begin{vmatrix} x+1 & w & w^2 \\ w & x+w^2 & 1 \\ w^2 & 1 & 1+w^2 \end{vmatrix} = 0$$

$R_1 \rightarrow R_1 + R_2 + R_3$

$$\begin{vmatrix} x+1+w+w^2 & 1+w+w^2+x & 1+w^2+w+x \\ w & x+w^2 & 1 \\ w^2 & 1 & 1+w^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & x & x \\ w & x+w^2 & 1 \\ w^2 & 1 & x+w \end{vmatrix} = 0$$

$$x\begin{vmatrix} 1 & 1 & 1 \\ w & x+w^2 & 1 \\ w^2 & 1 & x+w \end{vmatrix} = 0$$

$c_2 \rightarrow c_2 - c_1, c_3 \rightarrow c_3 - c_1$

$$x\begin{vmatrix} 1 & 1 & 1 \\ w & x+w^2-w & 1-w \\ w^2 & 1-w^2 & x-w^2+w \end{vmatrix} = 0$$

$x\{(x+w^2-w)-(1-w)(1-w^2)\}=0$

$\therefore \Rightarrow x = 0$

Q.54 If $\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & a \\ c-a & -a & 0 \end{vmatrix}$, then Δ is equal to

(A) (a+b) (b+c) (c+a) **(B)** bc + ca + ab
(C) 2abc **(D)** none of these

Ans: D

$$\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & a \\ c-a & -a & 0 \end{vmatrix}$$

$= -(a-b)\{-a(c-a)\} + (a-c)\{-a(b-a)\}$

$= a(a-b)(c-a) - (a-c)(b-a)a$

$= (a-b).a[a - c + a - c]$

$= 0$

Q.55 If A is a skew-symmetric matrix and n is a positive integer, then A^n is

(A) a symmetric matrix.

(B) skew-symmetric matrix for even n only.
(C) diagonal matrix.
(D) symmetric matrix for even n only.

Ans: D

Q.56 The period of the function sin x + sin 2x + sin 3x is

(A) π **(B)** π/2
(C) π/3 **(D)** 2π

Ans: D

$\sin(x+2\pi)+\sin(2\pi+2x)+\sin(2\pi+3x)$

$= \sin x+\sin 2x+\sin 3x$

$\because f(x+\theta)=f(x)$ then f(x) is periodic to Θ

Q.57 The Laplace transform of $L\left(\frac{e^t}{\sqrt{t}}\right)$ is

(A) $\sqrt{\frac{\pi}{s-1}}$ **(B)** $\sqrt{\frac{\pi}{(s+1)}}$

(C) $\sqrt{\frac{\pi}{s^2-1}}$ **(D)** $\sqrt{\frac{\pi}{s^2+1}}$

Ans: A

$$\because L\left\{\frac{\pi^t}{\sqrt{t}}\right\}=L\left\{e^t.t^{-1/2}\right\}$$

$$\int_0^\infty e^{-st}.e^t.t^{-1/2}dt$$

$$\int_0^\infty e^{-(s-t)}.t^{-1/2}dt$$

Putting $(s-1)t=\theta \quad \Rightarrow t=\frac{\theta}{s-1}$

$(s-1)dt=d\theta$

$dt=\frac{d\theta}{s-1}$

$$=\frac{1}{s-1}\int_0^\infty e^{-\theta}.\left(\frac{\theta}{s-1}\right)^{-1/2}d\theta$$

$$=\frac{\sqrt{s-1}}{(s-1)}\int_0^\infty e^{-\theta}.(\theta)^{-1/2}d\theta$$

$$=\frac{1}{\sqrt{s-1}}\int_0^\infty e^{-\theta}.(\theta)^{\frac{1}{2}-1}d\theta$$

$$= \frac{1}{\sqrt{s-1}}\sqrt{\frac{1}{2}} = \sqrt{\frac{\pi}{s-1}}$$

Q.58 The solution of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$ is

(A) $C_1e^{2x} - C_2e^{3x} + \frac{e^{3x}}{3}$ **(B)** $C_1e^{2x} + C_2e^{4x} + \frac{e^{4x}}{4}$

(C) $C_1e^{2x} + C_2e^{3x} + \frac{e^{4x}}{2}$ **(D)** $C_1e^{2x} - C_2e^{3x} - \frac{e^{4x}}{2}$

Ans: C

$(D^2 - 5D + 6)y = e^{4x}$

$A.E. \quad m^2 - 5m + 6 = 0$

(m – 3(m – 2) = 0

m = 2, 3

$C.F. = C_1e^{2x} + C_2e^{3x}$

P.I. = $\frac{1}{D^2 - 5D + 6}e^{4x}$

$= \frac{1}{16 - 20 + 6}e^{4x}$

$= \frac{1}{2}e^{4x}$

Y = C.F. + P.I. = $C_1e^{2x} + C_2e^{3x} + \frac{1}{2}e^{4x}$

Q.59 If $-3 + ix^2y$ and $x^2 + y + 4i$ represent conjugate complex numbers then the value of x and y is

(A) $x = \pm 1, y = -4$. **(B)** $x = -4, y = \pm 1$.

(C) $x = -4, y = -1$. **(D)** $x = 1, y = 4$.

Ans: A

$-3 + ix^2y, \quad x^2 + y + 4i$

$Let\ A = -3 + ix^2y$ (1)

$B = \overline{A} = x^2 + y + 4i$ (2)

The conjugate of A is $\overline{A} = -3 - ix^2y$

But given $\overline{A} = x^2 + y + 4i$

$-3 - ix^2y = x^2 + y + 4i$

$x^2 + y = -3$ (3)

$x^2y = -4$ (4)

$x^2y + y^2 = -3y$ (5)

$-4 + y^2 = -3y$

$y^2 + 3y - 4 = 0$

$y^2 + 4y - y - 4 = 0$

$y(y+4) - 1(y+4) = 0$

$(y+4)(y-1) = 0$

y = -4, 1

if y = -4 then by Eq. (4)

$x^2(-4) = -4$

$x^2 = 1$

$x = \pm 1$

Q.60 Imaginary part of $\sin \bar{z}$ is

(A) – cos x cosh y **(B)** – cos x sinh y

(C) – sin x cosh y **(D)** – sin x sinh y

Ans: B

Imaginary point of $\sin \bar{z}$

$\sin(x - iy) = \sin x \cos iy - \cos x \sin iy$

$= \sin x \cosh y - i \cos x \sinh y$

Imaginary part = -cos x sin hy

Q.61 Three vectors $\overline{A}, \overline{B}, \overline{C}$ are coplanar, the value of their scalar triple product is

(A) 0 **(B)** 1

(C) –1 **(D)** i

Ans: A

Q.62 If θ is the angle between the vectors $\bar{a}$ and $\bar{b}$ such that $|\bar{a} \times \bar{b}| = |\bar{a} \cdot \bar{b}|$ then θ is

(A) 0° **(B)** 45°

(C) 120° **(D)** 180°

Ans: B

$|\bar{a} \times \bar{b}| = |\bar{a} . \bar{b}|$

$\bar{a}\bar{b} \sin\theta = \bar{a}\bar{b} \cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^0$

Q.63 The value of the determinant $\begin{vmatrix} 1989 & 1990 & 1991 \\ 1992 & 1993 & 1994 \\ 1995 & 1996 & 1997 \end{vmatrix}$ is

(A) 1 **(B)** 2

(C) –1 **(D)** 0

Ans: D

The value of $\begin{vmatrix} 1989 & 1990 & 1991 \\ 1992 & 1993 & 1994 \\ 1995 & 1996 & 1997 \end{vmatrix}$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_2$

is $\begin{vmatrix} 1989 & 1 & 1 \\ 1992 & 1 & 1 \\ 1995 & 1 & 1 \end{vmatrix} = 0$

as two columns are similar

Q.64 If the product of two eigen values of the matrix $\begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ is 16, then the third eigen value is

(A) 0 **(B)** 5
(C) 2 **(D)** –2

Ans: C
Since the product of two eigen value of the matrix is 16. check is by the options, the product of all the eigen value, should be equal to the value of the determinants.
In this question value of determinants is
6(9 – 1) + 2(-6 + 2) + 2(2 – 6)
48 – 8 – 8 = 48 – 16 = 32
Since two eigen value product = 16
Hence for product to be 32, third eigen value should be 2.

Q.65 If f(x) is defined in (0, L), then the period of f(x) to expand it as a half range sine series is
(A) L. **(B)** 0.
(C) 2L. **(D)** $L/2$.

Ans: C

Q.66 The inverse Laplace transform $L^{-1}\left(1/s^n\right)$ is possible only when n is
(A) 0 **(B)** –ve integer
(C) –ve rational number **(D)** +ve integer

Ans: D

Q.67 The differential equation of a family of circles having the radius r and centre on the x axis is

(A) $y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=r^2$ **(B)** $x^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=r^2$

(C) $\left(x^2+y^2\right)\left[1+\left(\frac{dy}{dx}\right)^2\right]=r^2$ **(D)** $r^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=x^2$

Ans: A

The eq. of family of circle, having radius r, and centre on the x axis is

$(x-h)^2+y^2=r^2$ (1)

$2(x-h)+2y\frac{dy}{dx}=0$ (2)

$(x-h)=-y\frac{dy}{dx}$ (3)

Putting the value from eq.(3) into the eq.(1)

$$y^2\left(\frac{dy}{dx}\right)^2+y^2=r^2$$

$$y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=r^2$$

Q.68 If y satisfies $y''-3y'+2y=e^{-t}$ with $y(0)=y'(0)=0$ then Laplace transform $L(y(t))$ is

(A) $\frac{1}{(s+1)(s+2)^2}$ **(B)** $\frac{1}{(s+1)(s-2)^2}$

(C) $\frac{1}{(s+1)^2(s-2)}$ **(D)** $\frac{1}{(s+1)^2(s+2)}$

Ans: Correct option is not available; however the solution is:

$y''-3y'+2y=e^{-t}$ *with* $y(0)=y'(0)=0$

$L(y'')-3L(y)+2L(y)=L(e^{-t})$

$$[s^2\bar{y}-sy(0)-y'(0)]-3[\overline{sy}-y(0)]+2\bar{y}=\frac{1}{s+1}$$

$$s^2\bar{y}-3s\bar{y}+2\bar{y}=\frac{1}{s+1}$$

$$\bar{y}(s^2-3s+2)=\frac{1}{s+1}$$

$$\bar{y}(s-2)(s-1)=\frac{1}{s+1}$$

Or solution is $\bar{y}=\frac{1}{(s^2-1)(s-2)}$

Ans is D; if y satisfies $y''-3y'+2y=e^{-t}$ with $y(0)=y'(0)=0$

Q.69 If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$, $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ then z_1z_2 is equal to

(A) $\left(\frac{r_1}{r_2}\right)\{\text{Cos}(\theta_1+\theta_2)+i\,\text{Sin}(\theta_1+\theta_2)\}$.

(B) $r_1r_2\{\text{Cos}(\theta_1+\theta_2)+i\,\text{Sin}(\theta_1+\theta_2)\}$.

(C) $r_1r_2\{\text{Cos}(\theta_1\theta_2)+i\,\text{Sin}(\theta_1\theta_2)\}$.

(D) $r_1r_2\{\text{Cos}(\theta_1-\theta_2)+i\,\text{Sin}(\theta_1-\theta_2)\}$.

Ans. B

$z_1 = r_1$ (Cos θ_1 + i Sin θ_1)

$z_2 = r_2$ (Cos θ_2 + i Sin θ_2)

$z_1\ z_2 = r_1\ r_2$ (Cos θ_1 + i Sin θ_1) (Cos θ_2 + i Sin θ_2)

$= r_1\ r_2$ [(Cos θ_1 Cos θ_2 - Sin θ_1 Sin θ_2) + i (Cos θ_1 Sin θ_2 +Cos θ_2 Sin θ_1)]

$= r_1\ r_2$ [Cos (θ_1 +θ_2) + i Sin (θ_1 +θ_2)]

Q.70 If ω is cube root of unity then $1+\omega+\omega^2$ is equal to

(A) 0. **(B)** 1.

(C) -1. **(D)** 3.

Ans. A

If ω is cube root of unity then we know that $1+\omega+\omega^2=0$

Q.71 The roots of $x^2-x-12=0$ are

(A) 2, 3. **(B)** 3, 2.

(C) 4, -3. **(D)** 4, 3.

Ans. C

Given x^2-x-12=0 $\Rightarrow$ (x-4) (x+3) = 0 $\Rightarrow$ x=4,-3

Q.72 If $A=\begin{pmatrix}1&0\\1&0\end{pmatrix}$ and $B=\begin{pmatrix}0&0\\1&1\end{pmatrix}$ then AB is equal to

(A) $\begin{pmatrix}0&0\\0&0\end{pmatrix}$. **(B)** $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

(C) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. **(D)** $\begin{pmatrix}1&1\\1&1\end{pmatrix}$.

Ans. A

Given

$$A=\begin{pmatrix}1&0\\1&0\end{pmatrix}, B=\begin{pmatrix}0&0\\1&1\end{pmatrix} \Rightarrow AB=\begin{pmatrix}1&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Q.73. If A and B are invertible matrices of the same size then $(AB)^{-1}$ is equal to

(A) AB. **(B)** BA.

(C) $B^{-1}A^{-1}$. **(D)** $A^{-1}B^{-1}$.

Ans. C
Given
$A^{-1}A = I$, $B^{-1}B = I$
Now $(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$ --......................---------------(1)
Also $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$-----------------(2)
from 1 and 2, we get $(AB)^{-1} = B^{-1}A^{-1}$

Q.74 If A and B are the points (3, 4, 5) and (6, 8, 9) then the vector $\overrightarrow{AB}$ is

(A) $3\vec{i} + 4\vec{j} + 4\vec{k}$. **(B)** $3\vec{i} + 4\vec{j}$.
(C) $3\vec{i} - 4\vec{j} - 4\vec{k}$. **(D)** $3\vec{i} - 4\vec{j}$.

Ans. A
Given A (3,4,5) and B (6,8,9)

$$\overrightarrow{AB} = \text{Position vector of B - Position vector of A} = 3i + 4j + 4k$$

Q.75 The function f (x) = Sin x is

(A) non periodic. **(B)** periodic with period π.
(C) periodic with period 2π. **(D)** periodic with period $\pi/2$.

Ans. C
We know that the function f(x) = Sin x is periodic and period is 2π

Q.76 The Laplace transform of Sinh (at) is

(A) $\dfrac{1}{s^2 - a^2}$. **(B)** $\dfrac{a}{s^2 - a^2}$.
(C) $\dfrac{s}{s^2 + a^2}$. **(D)** $\dfrac{s}{s^2 - a^2}$.

Ans. B
By definition

$$L[\text{Sinhat}] = \int_o^\infty e^{-st} \text{ Sinhat dt}$$

$$= \int_o^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right)dt$$

$$= \frac{1}{2}\left[\int_o^\infty e^{-(s-a)t} - e^{-(s+a)t}\right]dt = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s-a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2a}{s^2 - a^2}\right] = \frac{a}{s^2 - a^2}$$

PART – II

NUMERICALS

Q.1 If the complex numbers z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that $z_1{}^2 + z_2{}^2 + z_3{}^2 = z_1z_2 + z_2z_3 + z_3z_1$. **(7)**

Ans:
Given that Z_1, Z_2, Z_3 be the vertices of an equilateral triangle.

$\therefore \dfrac{Z_3 - Z_1}{Z_2 - Z_1} = e^{(i\pi/3)}$

i.e. $(Z_3 - Z_1) = (Z_2 - Z_1)e^{(i\pi/3)}$ ....(1)

And $(Z_1 - Z_2) = (Z_3 - Z_2)e^{(i\pi/3)}$ ....(2)

Dividing (1) by (2) we get

$\dfrac{Z_3 - Z_1}{Z_1 - Z_2} = \dfrac{Z_2 - Z_1}{Z_3 - Z_2}$

$\Rightarrow (Z_3 - Z_1).(Z_3 - Z_2) = (Z_2 - Z_1)(Z_1 - Z_2)$

$\Rightarrow Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z + Z_3Z_2$

Q.2 If the roots of $z^3 + iz^2 + 2i = 0$ represent vertices of a triangle in the Argand plane, then find area of the triangle. **(7)**

Ans:
$Z^3 + iZ^2 + 2i = 0$

Root of above equation are the vertices of Δ

i, -i+1, -i-1

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{vmatrix} = \frac{1}{2}(-4) = -2$$

Q.3 Reduce $1 - \cos\alpha + i\sin\alpha$ to the modulus amplitude form. **(7)**

Ans:
$1 - \cos\alpha + i\sin\alpha$

$r^2 = (1 - \cos\alpha)^2 + \sin^2\alpha$

$= 1 - 2\cos\alpha + 1$

$r = \sqrt{2(1 - \cos\alpha)} = 2\cos\left(\dfrac{\alpha}{2}\right)$

$$\theta = \tan^{-1}\left(\frac{\sin\alpha}{1-\cos\alpha}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right) = \frac{\pi}{2}-\frac{\alpha}{2}$$

Q.4 Prove that $(1+\cos\theta+i\sin\theta)^n + (1+\cos\theta-i\sin\theta)^n = 2^{n+1}\left(\cos\frac{\theta}{2}\right)^n \cos\frac{n\theta}{2}$. **(7)**

Ans:

L.H.S.= $(1+\cos\theta+i\sin\theta)^n + (1+\cos\theta-i\sin\theta)^n$

$$=\left(2\cos^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n + \left(2\cos^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$=2^n\cos^n\frac{\theta}{2}\left[\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n + \left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)^n\right]$$

$$=2^n\cos^n\frac{\theta}{2}\left[\cos n\frac{\theta}{2}+i\sin n\frac{\theta}{2}+\cos n\frac{\theta}{2}-i\sin n\frac{\theta}{2}\right]$$

$$=2^n\cos^n\frac{\theta}{2}.2\cos n\frac{\theta}{2}$$

$$=2^{n+1}\cos^n\frac{\theta}{2}.\cos n\frac{\theta}{2}$$

=R.H.S. Hence proved.

Q.5 If a square matrix A satisfies a relation $A^2 + A^{-1} = 0$. Prove that A^{-1} exists and that $A^{-1} = I + A$, I being an identity matrix. **(7)**

Ans:

Given that a square matrix A satisfies a relation $A^2 + A^{-1} = 0$. By Cayley Hamilton Theorem

$\Rightarrow A + I - A^{-1} = 0$

$\Rightarrow A^{-1} = A + I$

Thus A^{-1} Exists

Q.6 Show that any square matrix can be written as the sum of two matrices, one symmetric and the other anti-symmetric. **(7)**

Ans:

Let A be a square matrix

Now $(A + A^t)^t = A^t + (A^t)^t$

$= A^t + A$

$= A + A^t$ ………..(1) is a symmetric matrix

Also $(A - A^t)^t = A^t - A$

$= -(A - A^t)$ ……..(2) is a skew-symmetric

Also

$$A = \frac{1}{2}\left(A + A^t\right) + \frac{1}{2}\left(A - A^t\right)$$

= symmetric matrix + skew-symmetric (from (1) and (2))

Q.7 Show that x = 2 is one root of the determinant $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$, and find other two roots. **(6)**

Ans:

Given $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$, when x = 2, then $\begin{vmatrix} 2 & -6 & -1 \\ 2 & -6 & -1 \\ -3 & 4 & 4 \end{vmatrix} = 0$

As two rows are same

Thus x – 2 is a root of given equation.

Now calculate other two Roots

Applying $R_1 \to R_1 - R_2$

$$(X-2)\begin{vmatrix} 1 & 3 & 1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

$C_2 \to C_2 - 3C_1$

$C_3 \to C_3 + C_1$

$$(X-2)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -3x-6 & x-1 \\ -3 & 2x+9 & x-1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(x-1)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -3(x+2) & 1 \\ -3 & 2x+9 & 1 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(x-1)(-5x-15) = 0$

$\Rightarrow (x-2)(x-1)(x+3) = 0$

x = 1, x = 2, x = -3

Thus other Roots are 1, -3

Q.8 Show that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2\,abc\,(a+b+c)^3$. **(8)**

Ans:

To prove $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

L.H.S. = $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

Applying $c_1 \to c_1 - c_3, c_2 \to c_2 - c_3$ we get

$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$

$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c^2 - a - b & (a+b)^2 \end{vmatrix}$

$\Rightarrow R_3 \to R_3 - (R_2 + R_1)$

$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$

$C_1 \to C_1 + \frac{1}{a} C_3$

$C_2 \to C_2 + \frac{1}{b} C_3$

$= (a+b+c)^2 \begin{vmatrix} b+c & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix} = 2abc(a+b+c)^3$

Q.9 If $\vec{a}$ and $\vec{b}$ be any two vectors, then show that

(i) $\left(\vec{a} + \vec{b}\right).\left(\vec{a} - \vec{b}\right) = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2$.

(ii) $\left|\vec{a} + \vec{b}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\,\vec{a} \cdot \vec{b}$. **(7)**

Ans:

(i) LHS = $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = \vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}\vec{b}$

$= |\vec{a}|^2 - \vec{a}.\vec{b} + \vec{a}.\vec{b} - |\vec{b}|^2$ $\{ \quad \vec{a}.\vec{b} = \vec{b}.\vec{a} \quad \}$

$|a|^2 - |\vec{b}|^2$ Hence Proved

(ii)

L.H.S = $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}\vec{b}$

$= |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{a}.\vec{b} + |\vec{b}|^2$

$= |\vec{a}| + 2\vec{a}.\vec{b} + |\vec{b}|^2$ Hence Proved

Q.10 Forces $\overrightarrow{F_1}, \overrightarrow{F_2}, \overrightarrow{F_3}$ of magnitudes 5, 3, 1 units respectively, act in the directions $6i + 2j + 3k, 3i - 2j + 6k, 2i - 3j - 6k$ respectively on a particle. If the particle is displaced from the point $(2,-1,-3)$ to the point $(5,-1,1)$, find the work done by the resultant force. **(7)**

Ans:

Force $\vec{f} = 5\vec{f}_1 + 3\vec{f}_2 + \vec{f}_3$

$= 5(6\hat{i} + 2\hat{j} + 3\hat{k}) + 3(3\hat{i} - 2\hat{j} + 6\hat{k}) + (2\hat{i} - 3\hat{j} - 6\hat{k})$

$\vec{f} = 41\hat{i} + \hat{j} + 27\hat{k}$

$\vec{d} = 3\hat{i} + 4\hat{k}$

$W = f.d = (41\hat{i} + \hat{j} + 27\hat{k}).(3\hat{i} + 4\hat{k})$

= 123 + 108

= 231

Q.11 Verify that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies its characterstic equation $x^2 - 3x - 7 = 0$ and then find A^{-1}. **(6)**

Ans:

$A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, $A.A = A^2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$

Characteristic Equation $= x^2 - 3x - 7 = 0$

By Clayey Hamilton theorem $A^2 - 3A - 7I = 0$

Now we have $A^2 - 3A - 7I$

$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

This verifies the characteristic equation.

Now $A^2 - 3A - 7I = 0$

Multiplying by A^{-1}

$A - 3I - 7A^{-1} = 0$

$\Rightarrow 7A^{-1} = A - 31$

$= \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$7A^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

Q.12 Test for the consistency and solve the system of equations.

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$. **(8)**

$7x + 2y + 10z = 5$

Ans:

Test for consistency

$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$, AX=B

Let $C = [A : B] = \begin{bmatrix} 5 & 3 & 7:4 \\ 3 & 26 & 2:9 \\ 7 & 2 & 10:5 \end{bmatrix}$

$R_2 \to 5R_2 - 3R_1$

$R_3 \to 5R_3 - 7R_1$

$= \begin{pmatrix} 5 & 3 & 7 & :4 \\ 0 & 121 & -11 & :33 \\ 0 & -11 & 1 & :-3 \end{pmatrix}$

$R_3 \to 11R_3 + R_2$

$= \begin{pmatrix} 5 & 3 & 7 & :4 \\ 0 & 121 & -11 & :33 \\ 0 & 0 & 0 & :0 \end{pmatrix}$

Now R(A) = R(C) = 2 < 3

System is consistent but infinity many solution.

Z = k, 11y – Z = 3

$Y = \dfrac{3+k}{11}$,

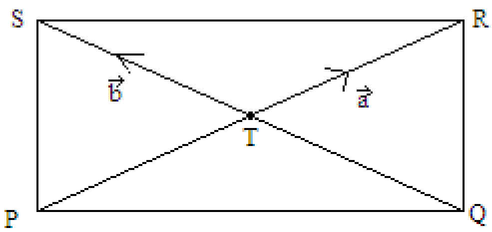
5x + 3y + 7z = 4

$\Rightarrow X = \dfrac{-16k+7}{11}$

Q.13 Show that the area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$. **(7)**

Ans:

Let PQRS be a parallelogram with diagonal $P\vec{R} = \vec{a}$ and $Q\vec{S} = \vec{b}$ they intersect at T



$\therefore \quad P\vec{Q} = P\vec{T} + T\vec{Q} = P\vec{T} - Q\vec{T}$

$= \dfrac{\vec{a}}{2} - \dfrac{\vec{b}}{2} = \dfrac{(\vec{a}-\vec{b})}{2}$

$P\vec{S} = P\vec{T} + T\vec{S} = \dfrac{\vec{a}}{2} + \dfrac{\vec{b}}{2} = \dfrac{(\vec{a}+\vec{b})}{2}$

Area of parallalogram PQRS $= \left|P\vec{Q}\times P\vec{S}\right|$

$= \left|\frac{1}{2}(\vec{a}-\vec{b})\times\frac{1}{2}(\vec{a}+\vec{b})\right|$

$= \frac{1}{4}\left|(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})\right|$

$= \frac{1}{4}\left|\vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}\right|$

$= \frac{1}{4}\left|(\vec{a}\times\vec{b}) + (\vec{a}\times\vec{b})\right| \quad (\therefore \vec{a}\times\vec{a} = 0), \quad (\vec{b}\times\vec{b} = 0) \textit{ and } \vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$

$= \frac{1}{4}\left|2(\vec{a}\times\vec{b})\right|$

$= \frac{1}{2}\left|\vec{a}\times\vec{b}\right|$ Hence proved.

Q.14 Find the area of the triangle whose vertices are $(3,-1,2)(1,-1,-3)(4,-3,1)$. **(7)**

Ans:

Let O be origin, $\overline{A} = (1,0,-1), \overline{B} = (2,1,5), \overline{C} = (0,1,2)$

$\overline{OA} = i - k, \overline{OB} = 2i + j + k, \overline{OC} = j + 2k, \overline{BC} = -2i - 3k, \overline{BA} = -i - j - 6k$

$\therefore$ Area of Δ ABC = $\frac{1}{2}\left|\overline{BC} \times \overline{BA}\right|$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ -2 & 0 & -3 \\ -1 & -1 & -6 \end{vmatrix}$$

$$= \frac{1}{2}\left|-3i - 9j + 2k\right|$$

$$= \frac{1}{2}\sqrt{94}$$

Q.15 Find a Fourier series that represents the periodic function f (x) = $x - x^2, \ -\pi \le x \le \pi$. **(14)**

Ans:

$f(x) = x - x^2$

Let $f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$ …………..(1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= -\frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx dx$$

$$= \frac{2}{\pi}\int_{-1}^{\pi} x^2 \cos nx dx \qquad (\because x\cos nx \text{ is odd function})$$

$$= -\frac{4}{n^2}(-1)^n$$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx dx \qquad (\because x^2 \sin nx \text{ is odd function})$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx$$

$$= \frac{-2}{n}(-1)^n$$

Putting value of a_0, a_n, b_n in (1) we get

$$x - x^2 = \frac{-\pi^2}{3} + 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - - -\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - - -\right)$$

Q.16 Find the Laplace transform of $\left\{\frac{1-e^t}{t}\right\}$. **(7)**

Ans:

$$L\left\{\frac{1-e^t}{t}\right\}$$

Now we have $L\{1-e^t\} = \frac{1}{s} - \frac{1}{s-1} = f(s)$

$$L\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty f(x)ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds$$

$= \log\frac{s-1}{s}$ Ans.

Q.17 Find the inverse Laplace transform of $\left\{\frac{s^2}{(s-2)^2}\right\}$. **(7)**

Ans:

$$L^{-1}\left[\frac{s^2}{(s-2)^2}\right]$$

$$= L^{-1}\left[\frac{(s-2+2)^2}{(s-2)^2}\right]$$

$$= L^{-1}\left[\frac{(s-2)^2 + 4 + 4(s-2)}{(s-2)^2}\right]$$

$$= L^{-1}[1] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right] + 4L^{-1}\left[\frac{1}{s-2}\right]$$

$= \delta(t) + 4e^{2t}t + 4e^{2t}$ Ans.

Q.18 Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^x$. **(7)**

Ans:

Solve $(D^2 + 5D + 6)y = e^x$

A.E., $m^2 + 5m + 6 = 0$

m = -2, -3

C.F = $C_1e^{-2x} + C_2e^{-3x}$

$$P.I = \frac{1}{D^2+5D+6}e^x = \frac{1}{12}e^x$$

$$Y = C.F + P.I = C_1e^{-2x} + C_2e^{-3x} + \frac{1}{12}e^x$$

Q.19 Use Laplace transform method to solve $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, if x = 2 and $\frac{dx}{dt} = -1$ at t = 0. **(7)**

Ans:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$$

Taking Lapalace transformation on both sides

$$\left(s^2-2s+1\right)x(s) - 2s + 1 + 4 = \frac{1}{s-1}$$

$$\Rightarrow \left(s^2-2s+1\right)x(s) = \frac{1}{s-1} + 2s - 5$$

$$\Rightarrow x(s) = \frac{2s^2-7s+6}{(s-1)^3}$$

$$\Rightarrow x(s) = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$x = 2L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{(s-1)^2}\right) + L^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$= 2e^t - 3t.e^t + \frac{t^2.e^t}{2!}$$

$$x = 2e^t + \frac{t^2e^t}{2} - 3te^t$$

Q.20 Express $\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$ in the form x+iy. **(8)**

Ans:

$$\frac{(\cos\theta+i\sin\theta)^8}{(\sin\theta+i\cos\theta)^4} = \frac{(\cos\theta+i\cos\theta)^8}{(-i^2\sin\theta+i\cos\theta)^4} = \frac{(\cos\theta+i\sin\theta)^8}{i^4(\cos\theta-i\sin\theta)^4} = \frac{(\cos\theta+i\sin\theta)^8}{(\cos\theta-i\sin\theta)^4}$$

$$= (\cos\theta+i\sin\theta)^8.(\cos\theta-i\sin\theta)^{-4}$$

$$= (\cos\theta+i\sin\theta)^{12} = \cos 12\theta + i\sin 12\theta$$

Q.21 Write down all the values of $(1+i)^{1/4}$. **(8)**

Ans:

Let 1 + I = r($\cos\theta + i\sin\theta$)

$r\cos\theta = 1, \quad r\sin\theta = 1$

$r = \sqrt{2}, \quad \theta = \frac{\pi}{4}$

$\therefore (1+i)^{1/4} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{1/4}, \qquad n = 0, 1, 2, 3$

$= (2)^{\frac{1}{8}}\left[\cos\frac{(8n\pi+\pi)}{16} + i\sin\frac{(8n\pi+\pi)}{16}\right], \qquad n = 0, 1, 2, 3$

$= 2^{1/8}\left[\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right], 2^{1/8}\left[\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right], 2^{1/8}\left[\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right], 2^{1/8}\left[\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right]$

Q.22 Using vector method prove that the altitudes of a triangle are concurrent. **(8)**

Ans:

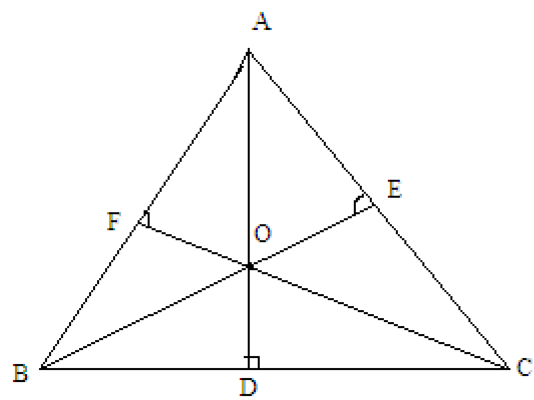
Let ABC be any angle

Draw AD ⊥ BC and BE ⊥ AC

Let AD and BE intersect at O. Join CO

We shall prove that CF ⊥ AB

Let ā, $\bar{b}$, $\bar{c}$ be the position vector of A, B, C respectively with O.



$AO \perp BC \Rightarrow \vec{AO}.\vec{BC} = 0$

$\Rightarrow -\bar{a}.(\bar{c} - \bar{b}) = 0$

$\Rightarrow \bar{a}.\bar{b} - \bar{a}.\bar{c} = 0 \qquad$ ....(1)

Also $BO \perp AC \Rightarrow -\bar{b}.(\bar{a} - \bar{c}) = 0$

$\Rightarrow \bar{b}.\bar{c} - \bar{b}.\bar{a} = 0 \qquad$ ....(2)

Adding (1) and (2) we get,

$(\bar{a}.\bar{b}) - (\bar{a}.\bar{c}) + (\bar{b}.\bar{c}) - (\bar{b}.\bar{a}) = 0$

$\Rightarrow \vec{AB}.\vec{OC} = 0 \quad \Rightarrow \vec{AB} \perp \vec{OC}$

$\Rightarrow AB \perp CF$

Hence altitude of a triangle is concurrent.

Q.23 Find a unit vector perpendicular to the plane of vectors $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + 2\vec{k}$. **(8)**

Ans:

$\vec{a} = 2\hat{i} + \hat{j} - \hat{k} \quad , \quad \vec{b} = \hat{i} - \hat{j} + 2\hat{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i} - 5\hat{j} - 3\hat{k}$$

$|a \times b| = \sqrt{35}$

$\therefore$ Unit vector perpendicular to $\bar{a}$ & $\bar{b}$

$$= \pm\frac{\bar{a}\times\bar{b}}{|a\times b|} = \pm\frac{1}{\sqrt{35}}\left(\hat{i}-5\hat{j}-3\hat{k}\right)$$

Q.24 Prove that $\left(\vec{b}\times\vec{c}\right)\cdot\left(\vec{a}\times\vec{d}\right)+\left(\vec{c}\times\vec{a}\right)\cdot\left(\vec{b}\times\vec{d}\right)+\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=0$ **(8)**

Ans:

$(\bar{b}\times\bar{c}).(\bar{a}\times\bar{d})+(\bar{c}\times\bar{a}).(\bar{b}\times\bar{d})+(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})$

$$\Rightarrow (\bar{b}\times\bar{c}).(\bar{a}\times\bar{d}) = \begin{vmatrix} b.a & b.d \\ c.a & c.d \end{vmatrix} = (b.a)(c.d)-(c.a)(b.d)$$

Now $(\bar{c}\times\bar{a}).(\bar{b}\times\bar{d}) = \begin{vmatrix} c.b & c.d \\ a.b & a.d \end{vmatrix} = (c.b)(a.d)-(a.b)(c.d)$

And $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = \begin{vmatrix} a.c & a.d \\ b.c & b.d \end{vmatrix} = (a.c)(b.d)-(b.c)(a.d)$

Adding equation1, equation 2 & equation 3 we get

$(\bar{b}\times\bar{c}).(\bar{a}\times\bar{d})+(\bar{c}\times\bar{a}).(\bar{b}\times\bar{d})+(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})$

= (b.c)(c.d) - (c.a)(b.d) + (c.b)(a.d) – (a.b)(c.d) - + (a.c)(b.d) – (b.c)(a.d)

$(\therefore a.b = b.a, c.d = d.c, c.a = a.c)$

= 0. Hence proved.

Q.25 Find the angle between two vectors $\vec{a}$ and $\vec{b}$ if $\left|\vec{a}\times\vec{b}\right| = \vec{a}\cdot\vec{b}$. **(8)**

Ans:

Let Angle between $\bar{a}$ and $\bar{b}$ be Θ

given $\left|\bar{a}\times\bar{b}\right| = a.b$

$\Rightarrow \left|\bar{a}\right|\left|\bar{b}\ \text{l}\sin\theta\right| = a.b$

$\Rightarrow \left|\bar{a}\right|\left|\bar{b}\right|\sin\theta = \bar{a}.\bar{b}$

$\Rightarrow \sin\theta = \dfrac{\bar{a}.\bar{b}}{\left|\bar{a}\right|\left|\bar{b}\right|}$

$\Rightarrow \tan\theta = 1 \qquad \Rightarrow \theta = \dfrac{\pi}{4}$

Q.26 Let A be a square matrix. Prove that A can be written the sum of a symmetric and a skew-symmetric matrix. **(8)**

Ans:

Let A be a square matrix

Let $A = \frac{1}{2}\left(A+A^t\right)+\frac{1}{2}\left(A-A^t\right)$

Now $\left(A+A^t\right)^t = A^t+\left(A^t\right)^t$

$= A^t + A$

$= A + A^t$ is a symmetric matrix $(\because A^t = A)$

Also $\left(A - A^t\right)^t = A^t - A = -\left(A - A^t\right)$ is skew-symmetric

Thus A = symmetric matrix + skew-symmetric.

Q.27 State Cayley Hamitton theorem and use it to find the inverse of $A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$, if the inverse exists. **(8)**

Ans:

Every square matrix satisfying its characteristic Equation.

$$|A - \lambda I| = 0 \qquad I,e \begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 4-\lambda & 5 \\ 0 & -6 & -7-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + 2\lambda^2 - \lambda - 20 = 0$$

By using Cayley-Hamilton Theorem

$$A^3 + 2A^2 - A - 20I = 0$$

$$A^2 + 2A - I - 20A^{-1} = 0$$

$$\Rightarrow 20A^{-1} = A^2 + 2A - I$$

$$\Rightarrow 20A^{-1} = \begin{bmatrix} 1 & 6 & 6 \\ 15 & -14 & -18 \\ -18 & 18 & 19 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Q.28 Prove that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$. **(8)**

Ans:

$$\text{L.H.S} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2 - a^2 & b^3 - a^3 \\ 0 & c^2 - a^2 & c^3 - a^3 \end{vmatrix} \qquad R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$= (b-a(c-a)\begin{vmatrix}1 & a^2 & a^3 \\ 0 & b+a & b^2+a^2+ab \\ 0 & c+a & c^2+a^2+ac\end{vmatrix} \qquad R_3 \to R_3 - R_2$$

$$= (b-a)(c-a)\begin{vmatrix}1 & a^2 & a^3 \\ 0 & b+a & b^2+a^2+ab \\ 0 & c-b & (c-b)(a+b+c)\end{vmatrix}$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix}1 & a^2 & a^3 \\ 0 & b+a & b^2+a^2+ab \\ 0 & 1 & a+b+c\end{vmatrix}$$

= (b – a)(c – a)(c – b)(ab + bc + ca)

= R.H.S.

Q.29 Give condition under which we can find λ so that the following system of linear equations has a non-trivial solution.

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$(p_1 + \lambda q_1)x + (p_2 + \lambda q_2)y + (p_3 + \lambda q_3)z = 0 \qquad \textbf{(8)}$$

Ans:

Given system of equation

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$(p_1 + \lambda q_1)x + (p_2 + \lambda q_2)y + (p_3 + \lambda q_3)z = 0$ is homogenous. For non trivial solution.

R(A) = R(C) < n here n = 3

Obviously R(A) = R(C) = 2 i.e. $|A| = 0$

$$\begin{vmatrix}a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1+\lambda q_1 & p_2+\lambda q_2 & p_3+\lambda q_3\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}1 & \frac{b_1}{a_1} & \frac{c_1}{a_1} \\ a_2 & b_2 & c_2 \\ p_1+\lambda q_1 & p_2+\lambda q_2 & p_3+\lambda q_3\end{vmatrix} = 0$$

$R_2 \to R_2 - a_2R_1,\ R_3 \to R_3 - (p_1 + \lambda q_1)$

$$\Rightarrow\begin{vmatrix} 1 & \frac{b_1}{a_1} & \frac{c_1}{a_1} \\ 0 & b_2-\frac{b_1a_2}{a_1} & c_2-\frac{c_1a_2}{a_1} \\ 0 & (p_2+\lambda q_2)-\frac{b_1}{a_1}(p_1+\lambda q_1) & (p_3+\lambda q_3)-\frac{c_1}{a_1}(p_1+\lambda q_1) \end{vmatrix}=0$$

R(A) must be 2.

$\therefore\ (p_2+\lambda q_2)-\frac{b_1}{a_1}(p_1+\lambda q_1)=0$

and $(p_3+\lambda q_3)-\frac{c_1}{a_1}(p_1+\lambda q_1)=0$

Q.30 Find the Fourier series of the function defined by

$$f(x)=\begin{cases} x+\pi & : \ 0\le x\le\pi \\ -x-\pi & : \ -\pi\le x<0 \end{cases}$$ **(8)**

Ans:

$$f(x)=\begin{Bmatrix} x+\pi & 0\le x\le\pi \\ -x-\pi & -\pi\le x\le 0 \end{Bmatrix}$$

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)dx=\pi$$

$$a_0=\pi$$

Now $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nx dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nx dx$$

$$=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$a_n=\begin{cases} \frac{-4}{n^2\pi}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$= \frac{1}{\pi}\int_{-\pi}^{0}(-\pi - x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi + x)\sin nx dx$

$= \frac{2}{n}\left[1-(-1)^n\right]$

$= \begin{cases} \frac{4}{\pi}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$

Putting value of a_0, a_n and b_n in (1)

$\therefore$ Fourier series

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right] + 4\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\right]$$

Q.31 Find the Fourier series representing the function

$f(x) = x \qquad 0 < x < 2\pi$ **(8)**

Ans:

$f(x) = x, \qquad 0 < x < 2\pi$

Let Fourier series of f(x)

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx \ \ldots\ldots\ldots\ldots(1)$$

Where $a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)dx$

$= \frac{1}{\pi}\int_{0}^{2\pi} x dx = 2\pi$

Now, $a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx dx$

$= \frac{1}{\pi}\int_{0}^{2\pi} x.\cos nx dx$

$= 0$

Now, $b_n = \frac{1}{\pi}\int_{0}^{2\pi} x \sin nx dx$

$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$

$= -\frac{2}{n}$

$$\therefore x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

Q.32 If F(t) is piecewise continuous and satisfies $\left|F(t)\right| \le Me^{at}$ for all $t \ge 0$ and for some constants a and M then

$$L\left\{\int_0^t F(x)dx\right\} = \frac{1}{s}L\{F(t)\}, (s > 0, s > a) \qquad \textbf{(8)}$$

Ans:

We are given $\left|F(t)\right| \le Me^{at}$ ....(1)

Without loss of generality, assume that a is positive.

Let $G(f) = \int_0^t F(x)dx$

Then G(t) is continuous.

Also $\left|G(t)\right| \le \int_0^t \left|F(x)\right|dx \le \int_0^t Me^{ax}dx$

$\therefore \left|G(t)\right| \le \frac{M}{a}\left(e^{at} - 1\right), a > 0$ ....(2)

Now $G'(t) = F(t)$ except for points where F(t) is discontinuous.

$\therefore G'(t)$ is piece-wise continuous on each finite interval.

We know that if F(t) is continuous for all t ≥ 0 and of experimental order a as, $t \to \infty$ and if $F'(t)$ is of class A, then

$L(F'(t)) = pL\{F(t)\}$ ----------F(o)

Therefore $L(G'(t)) = pL\{(t)\}$ ....G(o)

$= pL\{(t)\}$ as G(o)=0

Q.33 Define Inverse Laplace Transform of a function F(t). Prove that

$$L^{-1}\left\{\frac{1}{s^3+1}\right\} = \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!} + \ldots\ldots\ldots \qquad \textbf{(8)}$$

Ans:

$$\frac{1}{p^3+1} = \frac{1}{p^3}\left(1 + 1/p^3\right)^{-1}$$

$$= \frac{1}{p^3}\left[1 - \frac{1}{p^3} + \frac{1}{p^6} - \frac{1}{p^9} + \frac{1}{p^{12}} - - -\right]$$

$$= \frac{1}{p^3} - \frac{1}{p^6} + \frac{1}{p^9} - \frac{1}{p^{12}}$$

$$\therefore L^{-1}\left\{\frac{1}{p^3+1}\right\} = \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!}$$

Q.34 Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=\sin 2x$. **(8)**

Ans:

The given equation is

$(D^2+3D+L)y=\sin 2x$

Auxiliary equation is

$D^2+3D+L=0$

$\therefore D=-1,-2$

C.F = $C_1e^{-x}+C_2e^{-2x}$

$$\text{P.I} = \frac{1}{D^2+3D+2}\sin 2x$$

$$= \frac{1}{-4+3D+2}\sin 2x$$

$$= \frac{1}{3D-2}\sin 2x$$

$$= \frac{3D+2}{9D^2-4}\sin 2x$$

$$= -\frac{3D+2}{40}\sin 2x$$

$$= -\frac{1}{20}[3\cos 2x+\sin 2x]$$

Q.35 If a, b, c are real numbers such that $a^2+b^2+c^2=1$ and b + ic = (1 + a)z, where z is a complex number, then show that $\frac{1+iz}{1-iz}=\frac{a+ib}{1+c}$. **(8)**

Ans:

$$a^2+b^2+c^2=1; \quad \frac{b+ic}{1+a}=z; \quad \frac{b-ic}{1+a}=\bar{z}$$

$$z+\bar{z}=\frac{2b}{1+a}; \; z-\bar{z}=\frac{2ic}{1+a}; \; z\bar{z}=\frac{b^2+c^2}{(1+a)^2}=\frac{1-a}{1+a}$$

$$\text{Now} \quad \frac{1+iz}{1-iz}=\frac{1+iz}{1-iz}.\frac{1+i\bar{z}}{1+i\bar{z}}=\frac{1+i(z+\bar{z})-z\bar{z}}{1-i(z-\bar{z})+z\bar{z}}$$

$$=\frac{1+\frac{2bi}{1+a}-\frac{1-a}{1+a}}{1+\frac{2c}{1+a}+\frac{1-a}{1+a}}=\frac{1+a+2bi-1+a}{1+a+2c+1-a}=\frac{a+ib}{1+c}$$

Q.36 Given that $z_1 + z_2 + z_3 = A$, $z_1 + z_2\omega + z_3\omega^2 = B$ and $z_1 + z_2\omega^2 + z_3\omega = C$, where ω is a cube root of unity. Express z_1, z_2, z_3 in terms of A, B, C and ω.

(8)

Ans:

$$z_1 + z_2 + z_3 = A$$
$$z_1 + z_2\omega + z_3\omega^2 = B$$
$$z_1 + z_2\omega^2 + z_3\omega = C$$

On adding, $3z_1 + z_2(1+\omega+\omega^2) + z_3(1+\omega+\omega^2) = A+B+C$

$$\Rightarrow \quad z_1 = \frac{A+B+C}{3}$$

Again, $z_1(1+\omega+\omega^2) + z_2(1+\omega^3+\omega^3) + z_3(1+\omega^4+\omega^2) = A + B\omega^2 + C\omega$

$$\Rightarrow \quad z_2 = \frac{A+B\omega^2+C\omega}{3}$$

Similarly $\quad z_3 = \dfrac{A+B\omega+C\omega^2}{3}$

Q.37 Show that for all real μ, $\cos(6\mu) = 32\cos^6(\mu) - 48\cos^4(\mu) + 18\cos^2(\mu) - 1$. **(8)**

Ans:

$$\cos(6\mu) + i\sin(6\mu) = (\cos\mu + i\sin\mu)^6$$
$$= \cos^6\mu + 6i\cos^5\mu\sin\mu - 15\cos^4\mu\sin^2\mu - 20i\cos^3\mu\sin^3\mu$$
$$+15\cos^2\mu\sin^4\mu + 6i\cos\mu\sin^5\mu - \sin^6\mu$$
$$\Rightarrow \quad \cos 6\mu = \cos^6\mu - 15\cos^4\mu(1-\cos^2\mu) + 15\cos^2\mu(1-\cos^2\mu)^2 - (1-\cos^2\mu)^3$$
$$= 32\cos^6\mu - 48\cos^4\mu + 18\cos^2\mu - 1.$$

Q.38 For any four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ prove that

$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right) = \left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right) - \left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right).$$

Hence prove that $\left(\vec{b}\times\vec{c}\right)\cdot\left(\vec{a}\times\vec{d}\right) + \left(\vec{c}\times\vec{a}\right)\cdot\left(\vec{b}\times\vec{d}\right) + \left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right) = 0$ **(8)**

Ans:

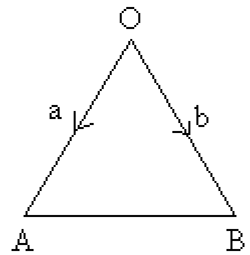
$$[\bar{a}\times\bar{b}].[\bar{c}\times\bar{d}] = \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix} = (\bar{a}.\bar{c})(\bar{b}.\bar{d}) - (\bar{a}.\bar{d}).(\bar{b}.\bar{c})$$

Adding the three relation we get

$(\bar{b}\times\bar{c}).(\bar{a}\times\bar{d})+(\bar{c}\times\bar{a}).(\bar{b}\times\bar{d})+(\bar{a}\times\bar{d})=0$

Q.39 In ΔOAB let $OA=\bar{a}$, $OB=\bar{b}$. Then find the vector representing AB and OM, where M is the midpoint of AB. **(4)**

Ans:

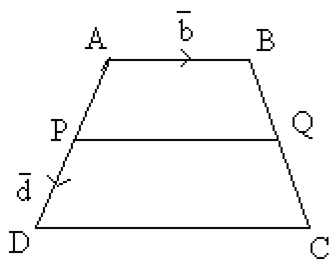


$AB = AO + AB = -\bar{a} + \bar{b} = \bar{b} - \bar{a}$

$OM = \dfrac{\bar{a}+\bar{b}}{2}$

Q.40 Prove that the straight line joining the mid-points of two non-parallel sides of a trapezium is parallel to the parallel sides and is half their sum. **(12)**

Ans:



let ABCD be the trapezium and let A be at origin

$AB=\bar{b},\ AD=\bar{d},\ AC=\bar{d}+t\,\bar{b}$

$AP=\dfrac{\bar{d}}{2}\quad ; \quad AQ=\dfrac{\bar{b}+(\bar{d}+t\,\bar{b})}{2}$

$\therefore\ PQ=\frac{1}{2}\bar{b}+\frac{1}{2}\bar{d}+\frac{1}{2}t\,\bar{b}-\frac{1}{2}\bar{d}=\frac{1}{2}(1+t)\,\bar{b}$

and $\dfrac{PQ}{AB}=\dfrac{1}{2}(1+t)=\dfrac{1}{2}\left(1+\dfrac{DC}{AB}\right)=\dfrac{1}{2}\left(\dfrac{AB+DC}{AB}\right)$

$\Rightarrow\quad PQ=\frac{1}{2}(AB+DC)$

i.e. PQ is parallel to AB and half the sum of parallel sides.

Q.41 For reals A, B, C, P, Q, R find the value of determinant

$$\begin{vmatrix}\cos(A-P) & \cos(A-Q) & \cos(A-R)\\ \cos(B-P) & \cos(B-Q) & \cos(B-R)\\ \cos(C-P) & \cos(C-Q) & \cos(C-R)\end{vmatrix}$$ **(8)**

Ans:

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A\cos P - \sin A\sin P & \cos A\cos Q - \sin A\sin Q & --- \\ \cos B\cos P - \sin B\sin P & \cos B\cos Q - \sin B\sin Q & --- \\ \cos C\cos P - \sin C\sin P & \cos C\cos Q - \sin C\sin Q & --- \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} = 0$$

Q.42 Using matrix method find the values of λ and μ so that the system of equations:

$2x - 3y + 5z = 12$

$3x + y + \lambda z = \mu$ has infinitely many solutions. **(8)**

$x - 7y + 8z = 17$

Ans:

$$[A \mid B] = \begin{bmatrix} 2 & -3 & 5 & 12 \\ 3 & 1 & \lambda & \mu \\ 1 & -7 & 8 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 8 & 17 \\ 3 & 1 & \lambda & \mu \\ 2 & -3 & 5 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 8 & 17 \\ 0 & 0 & \lambda-2 & \mu-7 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

$$If \quad \lambda = 2, \mu = 7 \quad [A \mid B] \sim \begin{bmatrix} 1 & -7 & 8 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 \end{bmatrix} \Rightarrow$$

x = 3 − z
y = z − 2
z = arbitrary

i.e. infinite solution.

Q.43 Solve the system of equations

$x + y + z = 6$

$x - y + 2z = 5$

$3x + y + z = 8$

by using inverse of a suitable matrix. **(8)**

Ans:

$$system \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$x + y + z = 6$$
$$-2y + z = -1$$
$$-z = -3$$
$$\Rightarrow \quad x = 1,\ y = 2,\ z = 3.$$

Q.44 Using Cayley-Hamilton theorem find A^3 for $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. **(8)**

Ans:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0.$$

$\therefore$ *By* Cayley - Hamilton theorem

$$A^2 - 4A - 5I = 0$$
$$\Rightarrow A^2 = 4A + 5I$$
$$A^3 = 4A^2 + 5A = 16A + 5A + 4.5.I$$
$$= 21A + 20I$$

$$\therefore\ A^3 = 21\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} 41 & 42 \\ 84 & 83 \end{bmatrix}$$

Q.45 State whether the function f(x) having period 2 and defined by

$$f(x) = 1 - x^2, -1 \le x \le 1$$

is even or odd. Find its Fourier Series. **(16)**

Ans:

$f(x) = 1 - x^2$ *is an even function*

$\therefore\ b_n = 0$

$$a_0 = 2\int_0^1 (1 - x^2)dx = \frac{4}{3}$$

$$a_n = 2\int_0^1 (1 - x^2)\cos n\pi x dx = \frac{2}{n\pi}\int_0^1 1 - x^2 d(\sin n\pi x)$$

$$= \frac{2}{n\pi}\left\{\left[(1 - x^2)\sin n\pi x\right]_0^1 + 2\int_0^1 x \sin n\pi x d(x)\right\}$$

$$= \frac{4}{n\pi}\int_0^1 x \sin n\pi x dx$$

$$= -\frac{4\cos n\pi}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$\therefore \ f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi x.$$

Q.46 Find the Laplace transform of $f(t) = e^{2t}t^2$. **(8)**

Ans:

Recall the first shift theorem

$$\alpha(e^{-at}f(t)) = F(s-a)$$

where $\alpha(f) = F(s)$.

$$\alpha(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$$

and so $\alpha(e^{-2t}f(t)) = \alpha(e^{-2t}t^2) = \dfrac{2}{(s-2)^3}$.

Q.47 Solve $(D^2 + D + 1)y = \cos 2x$. **(8)**

Ans:

$(D^2 + D + 1)y = \cos 2x$

A.E = $M^2 + M + 1 = 0$

$$\Rightarrow M = \frac{-1 \pm i\sqrt{1-4}}{2}$$

$$\Rightarrow \frac{-1 \pm i\sqrt{3}}{2}$$

C.F = $e^{-\frac{1}{2}x}\left\{C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right\}$

P.I = $\dfrac{1}{D^2 + D + 1}\cos 2x = \dfrac{1}{D-3}\cos 2x$

$$= \frac{D+3}{D^2-9}\cos 2x$$

$$= \frac{(D+3)\cos 2x}{D^2-9} = -\frac{1}{13}\{-2\sin 2x + 3\cos 2x\}$$

P.I = $-\dfrac{1}{13}\{3\cos 2x - 2\sin 2x\}$

$\Rightarrow$ Y = CF + P.I

$$Y = e^{-\frac{1}{2}x}\left\{C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right\} - \frac{1}{13}\{3\cos 2x - 2\sin 2x\} \qquad \text{Ans.}$$

Q.48 Find the Inverse Laplace transform for $L(s) = \dfrac{e^{-3s}}{(s-1)^4}$. **(8)**

Ans:

$$L^{-1}\left[\frac{1}{(s-1)^4}\right] = e^t L^{-1}\left[\frac{1}{s^4}\right] = e^t\,\frac{t^3}{3!} = \frac{1}{6}t^3 e^t$$

$$\therefore\quad L^{-1}\left[\frac{e^{-3s}}{(s-1)^4}\right] = \frac{1}{6}(t-3)^3 e^{t-3} \qquad t > 3$$

$$= 0 \qquad t < 3$$

Q.49 Solve the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3\sin x$$

given that y = -0.9 and $\dfrac{dy}{dx} = -0.7$, when x=0 **(8)**

Ans:

$(D^2 + 3D + 2)\,y = 3\sin x \quad m = -1, 2$

C.F = $C_1e^{-x} + C_2e^{-2x}$

P.I. = $\dfrac{1}{D^2+3D+2}(3\sin x) = 3\dfrac{1}{3D+1}\sin x = \dfrac{3}{10}(3D-1)\sin x$

$$y = C_1e^{-x} + C_2e^{-2x} - \frac{3}{10}(3\sin x - \cos x)$$

$$-0.9 = C_1 + C_2 + \frac{3}{10},\ -0.7 = -C_1 - 2C_2 - \frac{9}{10},\ y = -2.2e^{-x} + e^{-2x} - \frac{3}{10}(3\sin x - \cos x)$$

Q.50 Using the Laplace transform solve the differential equation
$f''(t) - 4f'(t) + 3f(t) = 1$ with initial conditions $f(0) = f'(0) = 0$. **(8)**

Ans:

$$Subsidiary\ \ equation \quad s^2F - 4sF + 3F = \frac{1}{s}$$

$$\Rightarrow \quad (s^2 - 4s + 3)F = \frac{1}{s}$$

$$\Rightarrow \quad \mathrm{F} = \frac{1}{\mathrm{s}(\mathrm{s}^2 - 4s + 3)}$$

$$= \frac{1}{s(s-1)(s-3)}$$

$$\frac{1}{\mathrm{s(s-1)(s-3)}} = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)}$$

$$\therefore \quad f(t) = \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t}$$

Q.51 If n is a positive integer, prove that $\left(\sqrt{3}+\mathrm{i}\right)^{\mathrm{n}} + \left(\sqrt{3}-\mathrm{i}\right)^{\mathrm{n}} = 2^{\mathrm{n}+1}\cos\frac{\mathrm{n}\pi}{6}$. **(8)**

Ans.

$\sqrt{3} + \mathrm{i} = \mathrm{r}(\cos\theta + i\sin\theta)$ …………(1)

$\mathrm{r}\cos\theta = \sqrt{3}$ …………(2)

$\mathrm{r}\sin\theta = 1$ …………(3)

from (2) and (3), r = 2, $\theta = \pi/6$

$$\therefore (\sqrt{3}+i)^n + (\sqrt{3}-i)^n = [r(\cos\theta + i\sin\theta)]^n + [r(\cos\theta - i\sin\theta)]^n$$

$$= r^n(\cos n\theta + i\sin n\theta) + r^n(\cos n\theta - i\sin n\theta)$$

$$= 2r^n \cos n\theta$$ ------>(4)

put the value of r and θ in eq n(y) we have

$$= 2.(2)^n.\cos\frac{n\pi}{6}$$

$$= 2^{n+1}\cos\frac{n\pi}{6}$$

Q.52 Find all the values of $\left(\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that the product of all these values is 1. **(8)**

Ans:

$$let\ \frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$$ ………..(1)

$$\therefore r\cos\theta = \frac{1}{2}$$ ……………(2)

$$r\sin\theta = \frac{\sqrt{3}}{2} \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

$$from\ (2)\ \&\ (3),\quad r = 1\ and\ \theta = \frac{\pi}{3}.$$

$$from\ (1),\quad \frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$

or

$$(\frac{1}{2} + i\frac{\sqrt{3}}{2})^{3/4} = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^{3/4}$$

$$= (\cos\pi + i\sin\pi)^{1/4}$$

$$= [\cos(2m\pi + \pi) + i\sin(2m\pi + \pi)]^{1/4},\ m = 0,1,2,3$$

$$= \cos\left(\frac{(2m\pi + \pi)}{4}\right) + i\sin\left(\frac{(2m\pi + \pi)}{4}\right)$$

where m = 0, 1, 2, 3.

$$\therefore The\ values\ are, \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$

$$\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right).$$

$\therefore$ The continued product of these roots

$$= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$$

$$= (\cos 4\pi + i\sin 4\pi)$$

$$= (\cos\pi + i\sin\pi)^4$$

$$= (-1)^4$$

$$= 1.$$

$\therefore\cos\pi = -1$ and $\sin\pi = 0$

Q.53 If the roots of $z^3 + iz^2 + 2i = 0$ represent vertices of a triangle in the Argand plane, then find area of the triangle. **(8)**

Ans:

Roots are z = i, -i + 1, -i -1,

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$$

$= 2.$

Q.54 Find the value of $(\vec{a} \times \vec{b}) \times \vec{c}$ if

$\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}, \vec{c} = \hat{i} - 2\hat{j} + 2\hat{k}$. **(8)**

Ans:

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=-\vec{c}\times\left(\vec{a}\times\vec{b}\right)$$

$$=-\left[\left(\vec{c}\,.\vec{b}\right)\vec{a}-\left(\vec{c}\,.\vec{a}\right)\vec{b}\right]$$

$$=\left[\left(\vec{c}\,.\vec{a}\right)\vec{b}-\left(\vec{c}\,.\vec{b}\right)\vec{a}\right]$$

$$\left(\bar{a}\times\bar{b}\right)=\begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

= i(-1) + 7j + 5k

$$\left(\bar{a}\times\bar{b}\right)\times\bar{c}=\begin{vmatrix} i & j & k \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix}$$

Now,

$$\left(\vec{c}\,.\vec{a}\right)=\left(\vec{i}-2\vec{j}+2\vec{k}\right).\left(3\vec{i}-\vec{j}+2\vec{k}\right)=9$$

$$\left(\vec{c}\,.\vec{b}\right)=\left(\vec{i}-2\vec{j}+2\vec{k}\right).\left(2\vec{i}+\vec{j}-\vec{k}\right)=-2$$

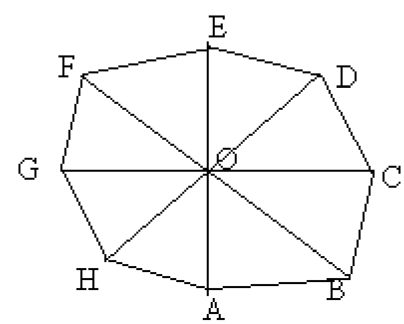
$$\therefore\quad \left(\vec{c}\,.\vec{a}\right)\vec{b}-\left(\vec{c}\,.\vec{b}\right)\vec{a}$$

$$=9\left(2\vec{i}+\vec{j}-\vec{k}\right)+2\left(3\vec{i}-\vec{j}+2\vec{k}\right)$$

$$=24\vec{i}+7\vec{j}-5\vec{k}$$

Q.55 Prove that the sum of all the vectors drawn from the centre of a regular octagon to its vertices is the zero vector. **(8)**

Ans:



Let ABCDEFGH be a regular octagon
And O the centre of this octagon, O is
the mid-point of diagonals AE, BF, CG and DH.
Now,

$$\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}+\vec{OE}+\vec{OF}+\vec{OG}+\vec{OH}$$

$$=\left(\vec{OA}+\vec{OE}\right)+\left(\vec{OB}+\vec{OF}\right)+\left(\vec{OC}+\vec{OG}\right)+\left(\vec{OD}+\vec{OH}\right)$$

$$=\left(\vec{OA}-\vec{OA}\right)+\left(\vec{OB}-\vec{OB}\right)+\left(\vec{OC}-\vec{OC}\right)+\left(\vec{OD}-\vec{OD}\right)$$

$$=\vec{O}+\vec{O}+\vec{O}+\vec{O}$$

$$=\vec{O}.$$

Q.56 Find the moment about the point $M(-2,4,-6)$ of the force represented in magnitude and position by $\vec{AB}$, where the point A and B have the co-ordinates $(1,2,-3)$ and $(3,-4,2)$ respectively. **(8)**

Ans:

$$\vec{AB}=(3i-4j+2k)-(i-2j+3k)$$
$$=2i-6j+5k$$

$$\vec{MA}=(i+2j-3k)-(-2i+4j-6k)$$
$$=3i-2j+3k$$

$$Moment=\vec{r}\times\vec{F}$$
$$=(3i-2j+3k)\times(2i-6j+5k)$$
$$=\begin{vmatrix} i & j & k \\ 3 & -2 & 3 \\ 2 & -6 & 5 \end{vmatrix}$$
$$=8i-9j-14k.$$

$$Magnitude\ of\ the\ moment=\sqrt{(8)^2+(-9)^2+(-14)^2}=\sqrt{341}$$

Q.57 Show that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}=1+a^2+b^2+c^2.$ **(8)**

Ans: Multiplying C_1, C_2, & C_3 by a, b and c respectively, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Taking out common a, b & c from R_1, R_2 and R_3 respectively.

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Now $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)$$

Q.58 Write the following system of equations in the matrix form AX = B and solve this for X by finding A^{-1}.

$$2x_1 - x_2 + x_3 = 4$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1 - 3x_2 - 2x_3 = 2$$

(8)

Ans:

Writing the given equations in matrix form, we have

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$So \quad AX = B$

$or\ X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -2 \end{vmatrix} = -5$$

$$\text{Adj.A} = \begin{bmatrix} 1 & -5 & -2 \\ 3 & -5 & -1 \\ -4 & 5 & 3 \end{bmatrix}$$

$$\text{A}^{-1} = \frac{\text{AdjA}}{|\text{A}|} = \begin{bmatrix} -\frac{1}{5} & 1 & \frac{2}{5} \\ -\frac{3}{5} & 1 & \frac{1}{5} \\ \frac{4}{5} & -1 & -\frac{3}{5} \end{bmatrix}$$

Now $X = \text{A}^{-1}\text{B}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & 1 & \frac{2}{5} \\ -\frac{3}{5} & 1 & \frac{1}{5} \\ \frac{4}{5} & -1 & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore \quad x_1 = 1, x_2 = -1, x_3 = 1$

Q.59 Using matrix methods, find the values of λ and μ so that the system of equations

$2x + 3y + 5z = 9$

$7x + 3y - 2z = 8$.

$2x + 3y - \lambda z = \mu$

has (i) unique solution and (ii) has no solution **(8)**

Ans: Ax = B

$$C = [A \mid B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

Applying $R_2 \to R_2 - \frac{7}{2} R_1$ and $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & -\frac{47}{2} \\ 0 & 0 & -\lambda - 5 & \mu - 9 \end{bmatrix}$$

(*i*) for unique solution C(A) = C(C) = 3

$$-\lambda - 5 \neq 0 \text{ and } \mu - 9 \neq 0$$

$$\lambda \neq -5, \mu \neq 9$$

(ii) For no solution

C(A) ≠ C(C)

If $-\lambda - 5 = 0$ C(A) = 2

And $\mu - 9 \neq 0$ C(C) = 3

For unique solution $\lambda \neq -5, \mu \neq 9$

For no solution $\lambda = -5, \mu \neq 9$

Q.60 Verify Cayley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Use Cayley Hamilton theorem to evaluate A^{-1} and hence solve the equations

x + 2y = 3

3x + y = 4 **(8)**

Ans:

$$A-\lambda I=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1-\lambda & 2\\ 3 & 1-\lambda\end{bmatrix}$$

$$\therefore \quad A-\lambda I=\begin{vmatrix}1-\lambda & 2\\ 3 & 1-\lambda\end{vmatrix}=\lambda^2-2\lambda-5=0$$

$\therefore$ The characteristic equation of A is $\lambda^2-2\lambda-5=0$

$$\therefore \quad A^2-2A-5I=\begin{bmatrix}7 & 4\\ 6 & 7\end{bmatrix}-2\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}-5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$$

$\therefore \quad A^2-2A-5I=0 \qquad (\therefore A^{-1}A=I)$

$5A^{-1}=A-2I$

$$A^{-1}=\frac{1}{5}\begin{bmatrix}-1 & 2\\ 3 & -1\end{bmatrix}$$

$AX=B$

$X=A^{-1}B$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-1 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}3\\ 4\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$\therefore x=1,\ y=1$

Q.61 Find the Fourier series for the functions

$$f(x)=\frac{1}{4}(\pi-x)^2,\ 0<x<2\pi \qquad \textbf{(16)}$$

Ans:

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\,dx$$

$$=\frac{1}{4\pi}\left[-\frac{1}{3}(\pi-x)^3\right]_0^{2\pi}=-\frac{1}{12\pi}\left[(-\pi^3)-\pi^3\right]=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\cos nx dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 . \quad \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2(-1)^2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi \cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi \cos o}{n^2} + 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \sin nxdx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 . \quad \frac{-\cos nx}{n} - 2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right)\right] = 0$$

$$\therefore \frac{1}{4}(\pi - x)^2 = \frac{1}{2}.\frac{\pi^2}{6} + \sum \frac{1}{n^2}\cos nx$$

$$= \frac{\pi^2}{12} + \frac{\cos x}{1} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + - - - - -$$

Q.62 Find the Laplace transform $L\left(te^{at} \sin at\right)$ **(8)**

Ans:

$$L(t) = \frac{1}{s}$$

$$\therefore L\left(te^{iat}\right) = \frac{1}{(s - ia)^2}$$

$$= \frac{1}{(s - ia)^2} \times \frac{(s + ia)^2}{(s + ia)^2}$$

$$= \frac{s^2 + 2ias + a^2 i^2}{(s^2 - i^2 a^2)^2}$$

$$= \frac{s^2 + 2ias - a^2}{(s^2 + a^2)^2}$$

$$\therefore L\{t(\cos at + \sin at)\} = -\frac{s^2 - a^2}{(s^2 + a^2)^2} + i\frac{2as}{(s^2 + a^2)^2}$$

Equating the imaginary parts, we have

$$L(t \sin at) = \frac{2as}{(s^2 + a^2)^2}$$

$$\therefore L\{e^{at}(t \sin at)\} = \frac{2a(s-a)}{\left[(s-a)^2 + a^2\right]^2}$$

$$= \frac{2a(s - a)}{(s^2 - 2as + a^2 + a^2)^2}$$

$$= \frac{2a(s - a)}{(s^2 - 2as + 2a^2)^2}$$

Q.63 Find the inverse Laplace transform $L^{-1}\left\{\frac{2s+1}{(s+1)(s^2+1)}\right\}$ **(8)**

Ans:

$$\frac{2s+1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs + C}{(s^2+1)}$$

$$(2s+1) = (A+B)s^2 + (B+C)s + (A+C)$$

$$\therefore A + B = 0,\ A + C = 1,\ B + C = 2$$

$$\therefore A = \frac{-1}{2}, B = \frac{1}{2}, C = \frac{3}{2}$$

$$\therefore L^{-1}\left\{\frac{2s+1}{(s+1)(s^2+1)}\right\} = L^{-1}\left\{\frac{-1}{2(s+1)}\right\} + L^{-1}\left\{\frac{(s+3)}{2(s^2+1)}\right\}$$

$$= \frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{3}{2}\sin t$$

Q.64 Solve the differential equation

$(D^2 + 9)y = \cos 3x$ **(8)**

Ans:

Auxiliary equation is

$$D^2 + 9 = 0$$
$$D^2 = -9$$
$$D = \pm 3i$$
$$C.F. = C_1 \cos 3x + C_2 \sin 3x.$$

$$P.I. = \frac{1}{(D^2 + 9)}(\cos 3x)$$

It is a case of failure.

$$\therefore P.I. = x\frac{1}{2D}\cos 3x$$
$$= \frac{x}{2}\int \cos 3x \, dx$$
$$= \frac{x}{2}\frac{\sin 3x}{3}$$
$$= \frac{x \sin 3x}{6}$$

Y = C.F. + P.I.

$$= C_1 \cos 3x + C_2 \sin 3x + \frac{x \sin 3x}{6}$$

Q.65 By using Laplace transform, solve the differential equation

$$\frac{d^2 y}{dt^2} + 9y = \cos 2t, \textit{ with initial conditions } \; y(0) = 1, \; y\left(\frac{\pi}{2}\right) = -1$$ **(8)**

Ans:

$$y'' + 9y = \cos 2t$$
$$(s^2\bar{y} - sy(0) - y'(0) + 9\bar{y}) = \frac{s}{s^2 + 4}$$
$$(s^2 + 9)\bar{y} - s(1) - A = \frac{s}{s^2 + 4} \qquad | \; y'(0) = A$$
$$\therefore \bar{y} = \frac{s}{(s^2 + 4)(s^2 + 9)} + \frac{s}{(s^2 + 9)} + \frac{A}{(s^2 + 9)}$$
$$\bar{y} = \frac{s}{5(s^2 + 4)} + \frac{4s}{5(s^2 + 9)} + \frac{A}{(s^2 + 9)}$$

Taking inverse Laplace transform

$$y = \frac{1}{5}L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{4}{5}L^{-1}\left(\frac{s}{s^2+9}\right) + A.L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t.$$

$$when\ \mathrm{x} = \frac{\pi}{2}\ \text{then y} = \text{-}1,$$

$$\therefore \text{-}1 = \frac{1}{5}\cos\pi + \frac{4}{5}\cos\frac{3\pi}{2} + \frac{\mathrm{A}}{3}\sin\frac{3\pi}{2}.$$

$$\frac{\mathrm{A}}{3} = 1 - \frac{1}{5} = \frac{4}{5}$$

$$A = \frac{12}{5}.$$

$$\therefore y = \frac{1}{5}\cos 2t + \frac{4}{5}(\cos 3\mathrm{t} + \sin 3\mathrm{t})$$

Q.66 A rigid body is spinning with angular velocity 27 radians per second about an axis parallel to $2\hat{\mathrm{i}} + \hat{\mathrm{j}} - 2\hat{\mathrm{k}}$ passing through the point $\hat{\mathrm{i}} + 3\hat{\mathrm{j}} - \hat{\mathrm{k}}$. Find the velocity of the point of the body whose position vector is $4\hat{\mathrm{i}} + 8\hat{\mathrm{j}} + \hat{\mathrm{k}}$. **(8)**

Ans:

$$\hat{w} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}\left(2\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$\vec{w} = w\hat{w} = 27.\frac{1}{3}\left(2\hat{i} + \hat{j} - 2\hat{k}\right) = 9\left(2\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$\vec{w} = 18\hat{i} + 9\hat{j} - 18\hat{k}$$

$$\vec{r} = \left(4\hat{i} + 8\hat{j} + \hat{k}\right) - \left(\hat{i} + 3\hat{j} - \hat{k}\right)$$

$$= 3\hat{i} + 5\hat{j} + 2\hat{k}$$

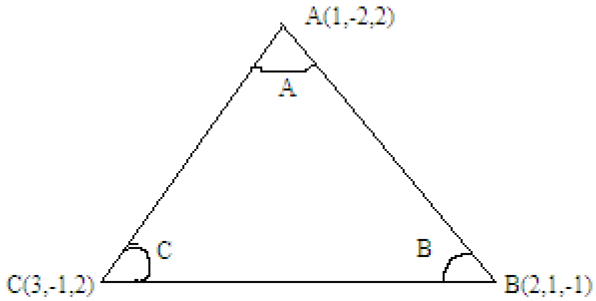
$$\vec{v} = \vec{w}\times\vec{r}$$

$$= 9\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & 5 & 2 \end{vmatrix}$$

$$= 9\left[12\hat{i} - 10\hat{j} + 7\hat{k}\right]$$

Q.67 Find the sides and angles of the triangle whose vertices are $\hat{\mathrm{i}} - 2\hat{\mathrm{j}} + 2\hat{\mathrm{k}}$, $2\hat{\mathrm{i}} + \hat{\mathrm{j}} - \hat{\mathrm{k}}$ and $3\hat{\mathrm{i}} - \hat{\mathrm{j}} + 2\hat{\mathrm{k}}$. **(8)**

Ans:



$\overline{AB} = B - A$

$= (2,1,-1) - (1,-2,2)$

$= \hat{i} + 3\hat{j} - 3\hat{k}$

$\overline{BC} = \overline{C} - \overline{B}$

$= (3,-1,2) - (2,1,-1)$

$= \hat{i} - 2\hat{j} + 3\hat{k}$

$\overline{CA} = \overline{C} - \overline{A}$

$= (3,-1,2) - (1,-2,2)$

$= 2\hat{i} + \hat{j}$

$$\cos C = \frac{\overline{AC}.\overline{BC}}{|\overline{AC}||\overline{BC}|} = \frac{(2\hat{i} + \hat{j}).(\hat{i} - 2\hat{j} + 3\hat{k})}{\sqrt{5}.\sqrt{14}} = 0$$

$\Rightarrow \cos C = 0$

$\Rightarrow C = \dfrac{\pi}{2}$

$$\text{Now } \cos B = \frac{\overline{BC}.\overline{AB}}{|BC||AB|} = \frac{(\hat{i} - 2\hat{j} + 3\hat{k})(\hat{i} + 3\hat{j} - 3k)}{\sqrt{14}.\sqrt{19}} = -\frac{14}{\sqrt{14 \times 19}}$$

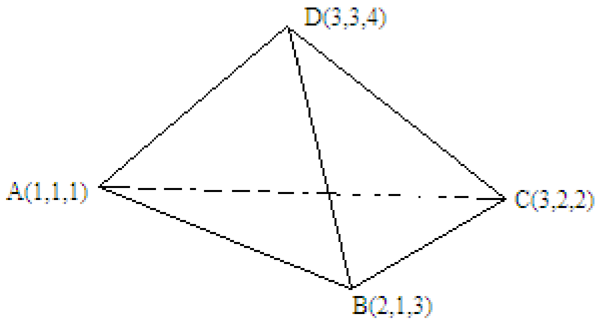
$$B = \cos^{-1}\left(\frac{-14}{\sqrt{14 \times 19}}\right)$$

$$\text{Now } \cos A = \frac{\overline{AC}.\overline{AB}}{|\overline{AC}||\overline{AB}|} = \frac{(2\hat{i} + j)(\hat{i} + 3\hat{j} - 3k)}{\sqrt{5}.\sqrt{19}} = \frac{9}{\sqrt{95}}$$

$$A = \cos^{-1}\left[\frac{5}{\sqrt{95}}\right]$$

Q.68 Find the volume of the tetrahedron formed by the point (1,1,1) (2,1,3) (3,2,2,), (3,3,4). **(8)**

Ans:



$$\overline{A_1} = \frac{1}{2}\left(\overline{DB} \times \overline{DC}\right), \quad \overline{A_2} = \frac{1}{2}\left(\overline{DC} \times \overline{DA}\right)$$

$$\overline{A_3} = \frac{1}{2}\left(\overline{DA} \times \overline{DB}\right), \quad \overline{A_4} = \frac{1}{2}\left(\overline{AC} \times \overline{AB}\right)$$

$$\overline{DB} = \overline{B} - \overline{D} = -\hat{i} - 2\hat{j} - \hat{k}$$

$$\overline{DC} = \overline{C} - \overline{D} = -\hat{j} - 2\hat{k}$$

$$\overline{DA} = \overline{A} - \overline{D} = -2\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\overline{AC} = \overline{C} - \overline{A} = 2\hat{i} + \hat{j} + \hat{k}$$

$$\overline{AB} = \overline{B} - \overline{A} = \hat{i} + 2\hat{k}$$

$$\overline{A_1} = \frac{1}{2}\left(\overline{DB} \times \overline{DC}\right) = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= \frac{1}{2}\left(3\hat{i} - 2\hat{j} + k\right) \Rightarrow |A_1| = \frac{1}{2}\sqrt{14}$$

$$\overline{A_2} = \frac{1}{2}\left(\overline{DC} \times \overline{DA}\right) = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & -2 \\ -2 & -2 & -3 \end{vmatrix}$$

$$= \frac{1}{2}\left(-\hat{i} + 4\hat{j} - 2k\right) \Rightarrow |A_2| = \frac{1}{2}\sqrt{21}$$

$$\overline{A_3} = \frac{1}{2}\left(\overline{DA} \times \overline{DB}\right) = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & -3 \\ -1 & -2 & -1 \end{vmatrix}$$

$$= \frac{1}{2}\left(-4\hat{i} + \hat{j} + 2\hat{k}\right) \Rightarrow |A_3| = \frac{1}{2}\sqrt{21}$$

$$\overline{A_4} = \frac{1}{2}\left(\overline{AC} \times \overline{AB}\right) = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

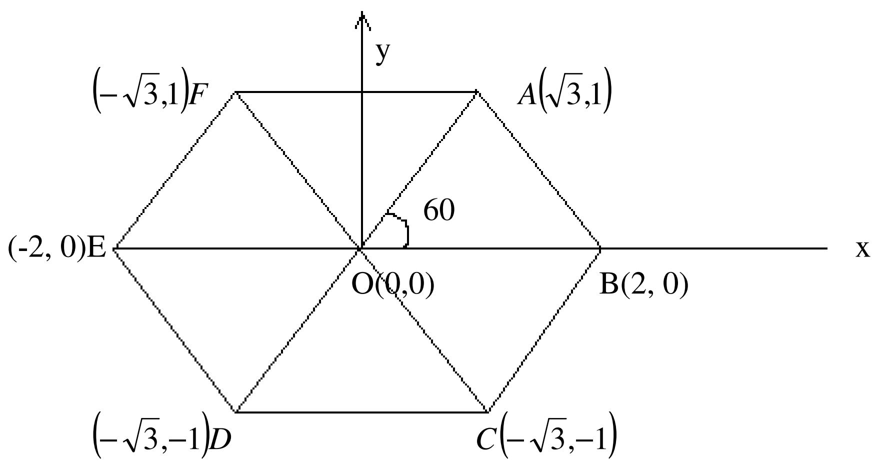
$$= \frac{1}{2}\left(2\hat{i} - 3\hat{j} - \hat{k}\right) \Rightarrow |A_4| = \frac{1}{2}\sqrt{14}$$

$\therefore$ Area of Trapezium = $A_1 + A_2 + A_3 + A_4$

$$= \sqrt{14} + \sqrt{21}$$
$$= \sqrt{7 \times 2} + \sqrt{7 \times 3}$$
$$= \sqrt{7}\left[\sqrt{2} + \sqrt{3}\right]$$

Q.69 The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + \mathrm{i}$ on the Argand diagram. Determine the other vertices. **(8)**

Ans:



$$A\left(\sqrt{3},1\right), \qquad \text{B(2,0)}, \qquad C\left(-\sqrt{3},-1\right)$$
$$D\left(-\sqrt{3},-1\right), \qquad \text{E(-2,0)}, \qquad F\left(-\sqrt{3},1\right)$$
$$\angle AOB = 60^0, \quad OA = 2. \quad \therefore OB = 2$$

Q.70 Prove that the general value of θ which satisfies the equation

$$(\cos\theta + \mathrm{i}\sin\theta)(\cos 2\theta + \mathrm{i}\sin 2\theta)\cdots(\cos n\theta + \mathrm{i}\sin n\theta) = 1 \text{ is } \frac{4m\pi}{n(n+1)},$$

where m is any integer **(8)**

Ans:

$$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)(\cos 3\theta + i\sin 3\theta) ---$$
$$-(\cos n\theta + i\sin n\theta) = 1$$
$$or \; \cos(\theta + 2\theta + --- + n\theta) + i\sin(\theta + 2\theta + --- + n\theta) = 1$$
$$\cos(1 + 2 + -- + n)\theta + i\sin(1 + 2 + -- + n)\theta = 1$$
$$\cos\left[\frac{n}{2}(n+1)\theta\right] + i\sin\left[\frac{n}{2}(n+1)\theta\right] = 1$$

Equating real part on both side

$\cos\frac{n}{2}(n+1)\theta = 1$

$\Rightarrow \frac{n}{2}(n+1)\theta = 2m\pi \pm 0$

$\Rightarrow n(n+1)\theta = 4m\pi$

$\theta = \frac{4m\pi}{n(n+1)}$

Q.71 Use De Moivre's theorem to solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$ **(8)**

Ans:

Given that $x^4 - x^3 + x^2 - x + 1 = 0$

Multiplying on both side by (x + 1)

$\Rightarrow x^5 + 1 = 0$

$\Rightarrow x = (-1)^{1/5}$

$\Rightarrow x = [\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{1/5}$

Putting n = 0,1,2,3,4

$\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right), \left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right), (\cos\pi + i\sin\pi),$

$\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right), \textit{and} \left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$

But $\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} = \cos\left(2\pi - \frac{3\pi}{5}\right) + i\sin\left(2\pi - \frac{3\pi}{5}\right)$

$= \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

Hence roots of $x^5 + 1 = 0$ are

$\left(\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}\right) \textit{and} -1$

But root → corresponding to (x + 1)

$\therefore$ Root of the equation $x^4 - x^3 + x^2 - x + 1 = 0$

$\left(\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}\right) \textit{and} \left(\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}\right)$

Q.72 Show that

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3\left(a^2+b^2+c^2+d^2+\lambda\right)$$ **(8)**

Ans:

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

$$= \frac{1}{abcd}\begin{vmatrix} a(a^2+\lambda) & ab^2 & ac^2 & ad^2 \\ a^2b & b(b^2+\lambda) & bc^2 & bd^2 \\ a^2c & b^2c & c(c^2+\lambda) & cd^2 \\ a^2d & b^2d & c^2d & d(d^2+\lambda) \end{vmatrix}$$

$$= \frac{abcd}{abcd}\begin{vmatrix} a^2+\lambda & b^2 & c^2 & d^2 \\ a^2 & b^2+\lambda & c^2 & d^2 \\ a^2 & b^2 & c^2+\lambda & d^2 \\ a^2 & b^2 & c^2 & d^2+\lambda \end{vmatrix}$$

$c_1 \to c_1 + c_2 + c_3 + c_4$

$$= (a^2+b^2+c^2+d^2+\lambda)\begin{vmatrix} a^2+\lambda & b^2 & c^2 & d^2 \\ 1 & b^2+\lambda & c^2 & d^2 \\ 1 & b^2 & c^2+\lambda & d^2 \\ 1 & b^2 & 1 & d^2+\lambda \end{vmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_3$

$$= (a^2+b^2+c^2+d^2+\lambda)\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix}$$

$$= (a^2+b^2+c^2+d^2+\lambda)\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= \lambda^3(a^2+b^2+c^2+d^2+\lambda)$$

R.H.S. hence proved.

Q.73 Express the following matrix as a sum of a symmetric matrix and a skew symmetric matrix.

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}.$$ **(8)**

Ans:

$$A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$$

$$A^t = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$$

Thus $A + A^t = \begin{vmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{vmatrix}$ and $A - A^t = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix}$

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

$$A = \begin{vmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} + \begin{vmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix}$$

A = symmetric + skew symmetric

Q.74 Find the values of λ, for which following system of equations has non-trivial solutions. Solve equations for all such values of λ.

$$(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$$
$$(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$$
$$2x + (3\lambda+1)y + 3(\lambda-1)z = 0$$

(8)

Ans:

AX = B

$\Rightarrow C = [A : B]$

$$A = \begin{bmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{bmatrix}$$

If system of equations has non-trivial solution then R(A) = R(C) < n = 3

$\therefore |A| = 0$

$$\Rightarrow \begin{bmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{bmatrix} = 0$$

$$R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 0 & -\lambda+3 & \lambda-3 \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{bmatrix} = 0$$

$$C_2 \to C_2 + C_3$$

$$\begin{bmatrix} 0 & 0 & \lambda-3 \\ \lambda-1 & 5\lambda+1 & \lambda+3 \\ 2 & 6\lambda-2 & 3\lambda-3 \end{bmatrix} = 0$$

$\Rightarrow (\lambda-3)[(\lambda-1)(6\lambda-2)-2(5\lambda+1)]=0$

$\Rightarrow (\lambda-3)[6\lambda^2-18\lambda]=0$

$\lambda = 0,3,3$

Putting $\lambda = 0$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 + 2R_1$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$R_3 \to R_3 + R_1$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad -x + y = 0$

$-3y + 3z = 0 \quad$ let $z = k_1$

$y = k_1, \quad x = k_1$ (infinite solution)

At $\lambda = 3$ $\quad A = \begin{bmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix}$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 - R_1$$

$$= \begin{bmatrix} 2 & 10 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad 2x + 10y + 6z = 0$$

Let $y = k_2, z = k_3$

$x = -5k_2 - 3k_3$.

Q.75 Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence evaluate the matrix equation $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$. **(8)**

Ans:

$|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By using Cayley-Hamilton Theorem

$A^3 - 5A^2 + 7A - 3I = 0$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 - 7A - 3I) + A^2 + A + I$

$= A^5.0 + A.0 + A^2 + A + I$

$= A^2 + A + I$

$$\because \quad A^2 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Q.76 Expand $f(x) = \sqrt{1-\cos x}$, $0 < x < 2\pi$ in a Fourier Series.

Hence evaluate $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$ **(16)**

Ans:

$f(x) = \sqrt{1-\cos x}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where, $f(x) = \sqrt{1-\cos x} = \sqrt{2}\sin\frac{x}{2}, 0 < x < 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) = \frac{1}{\pi} \times \sqrt{2}\int_0^{2\pi} \sin\frac{x}{2}\,dx$$

$$= \frac{1}{\pi} \times 2\sqrt{2}\int_0^{\pi} \sin dt = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx dx$$

$$= \frac{1}{2\pi}\times\sqrt{2}\int_0^{2\pi} 2\sin\frac{x}{2}\cos nx dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)x + \sin\left(\frac{1}{2}-n\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)} + \frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\left[-\frac{1}{\left(n+\frac{1}{2}\right)}\left\{\cos\left(n+\frac{1}{2}\right)2\pi - 1\right\} + \frac{1}{\left(n-\frac{1}{2}\right)}\left\{\cos\left(n-\frac{1}{2}\right)2\pi - 1\right\}\right]$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{2}{\left(n+\frac{1}{2}\right)} - \frac{2}{\left(n-\frac{1}{2}\right)}\right]$$

$$= \frac{4}{\sqrt{2}\pi}\left[\frac{1}{(2n+1)} - \frac{1}{(2n-1)}\right]$$

$$= \frac{-8}{\sqrt{2}\pi(4n^2-1)}$$

$$= \frac{8}{\sqrt{2}\pi(1-4n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \sin\frac{x}{2}\sin nx dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin\frac{x}{2}\sin nx dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\cos\left(\frac{1}{2}-n\right)x - \cos\left(\frac{1}{2}+n\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)x}{\left(n-\frac{1}{2}\right)} - \frac{\sin\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}}\right]_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\left[\left\{\sin\left(n-\frac{1}{2}\right)2\pi\right\}\frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}}\left\{\sin\left(n+\frac{1}{2}\right)2\pi\right\}\right]$$

$$= \frac{1}{\sqrt{2}\pi}\left[\sin(2n-1)\pi\frac{2}{2n-1} - \frac{2}{2n+1}\sin(2n+1)\pi\right]$$

$b_n = 0$

Thus the fourier series is

$$f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{(1-4n^2)\pi}\cos nx$$

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi}\left[1 - \frac{2}{3}\cos x - \frac{2}{15}\cos 2x - \frac{2}{35}\cos 3x\ldots\right]$$

Let x = 0,

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\left[\frac{1}{3} + \frac{1}{1.3.5} + \frac{1}{5.7} + \ldots\right]$$

$$\frac{2\sqrt{2}}{\pi} = \frac{4\sqrt{2}}{\pi}\left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\right]$$

Thus $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots = \frac{1}{2}$

Q.77 Simplify $\left(\frac{1+\text{Sin}\,\theta + \text{i}\,\text{Cos}\,\theta}{1+\text{Sin}\,\theta - \text{i}\,\text{Cos}\,\theta}\right)^n$. **(8)**

Ans:

$$\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n$$

$$= \left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n \qquad \text{putting } \theta = \frac{\pi}{2} - \phi$$

$$= \left[\frac{2\cos^2\frac{\phi}{2} + i2\sin\frac{\phi}{2}.\cos\frac{\phi}{2}}{2\cos^2\frac{\phi}{2} - i2\sin\frac{\phi}{2}.\cos\frac{\phi}{2}}\right]^n$$

$$= \left[\frac{\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}}{\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}}\right]^n$$

$$= \left(\cos\tfrac{\phi}{2} + i\sin\tfrac{\phi}{2}\right)^n.\left(\cos\tfrac{\phi}{2} - i\sin\tfrac{\phi}{2}\right)^{-n}$$

$$= \left(\cos\tfrac{\phi}{2} + i\sin\tfrac{\phi}{2}\right)^n.\left(\cos\tfrac{\phi}{2} + i\sin\tfrac{\phi}{2}\right)^n$$

$$= \cos n\phi + i\sin\phi$$

$$= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$$

Q.78 Find all the values of $(1+\mathrm{i})^{1/5}$. **(8)**

Ans:

Let $(1+i) = r(\cos\theta + i\sin\theta)$

$r = \sqrt{2}, \theta = \dfrac{\pi}{4}$

$$(1+i)^{1/5} = (\sqrt{2})^{1/5}\left[\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right]^{1/5}$$

$$= 2^{1/10}\left[\cos\left(\frac{8n\pi + n}{4}\right) + i\sin\left(\frac{8n\pi + n}{4}\right)\right]^{1/5}$$

$$= 2^{1/10}\left[\cos\left(\frac{8n\pi + n}{20}\right) + i\sin\left(\frac{8n\pi + n}{20}\right)\right]$$

Putting n = 0,1,2,3,4

$$= 2^{1/10}\left[\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right], 2^{1/10}\left[\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20}\right],$$

$$2^{1/10}\left[\cos\frac{17\pi}{20} + i\sin\frac{17\pi}{20}\right], 2^{1/10}\left[\cos\frac{25\pi}{20} + i\sin\frac{25\pi}{20}\right],$$

$$2^{1/10}\left[\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right]$$

Q.79 If Z_1 and Z_2 are two complex numbers, prove that $|Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2$ If and only if $\dfrac{Z_1}{Z_2}$ is purely imaginary. **(8)**

Ans:

Prove that

$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary

First assuming that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ and prove that

$\dfrac{z_1}{z_2}$ is purely imaginary

Given $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

Let $z_1 = (x_1 + iy_1),\ z_2 = (x_2 + iy_2)$,

$|(x_1 + x_2) + i(y_1 + y_2)|^2 = |(x_1 + iy_1)|^2 + |(x_2 + iy_2)|^2$

$\Rightarrow (x_1 + x_2)^2 + (y_1 + y_2)^2 = x_1^{\ 2} + y_1^{\ 2} + x_2^{\ 2} + y_2^{\ 2}$

$\Rightarrow x_1x_2 + y_1y_2 = 0$ (given)

Now we have

$$\frac{z_1}{z_2}=\frac{(x_1+iy_1)}{(x_2+iy_2)}\times\frac{(x_2-iy_2)}{(x_2-iy_2)}$$

$$=\frac{x_1x_2-ix_1y_2+iy_1x_2+y_1y_2}{x_2^{\;2}+y_2^{\;2}}$$

$$=\frac{x_1x_2+y_1y_2+i(y_1x_2-x_1y_2)}{x_2^{\;2}+y_2^{\;2}}$$

$$=\frac{0+i(y_2x_2-x_1y_2)}{x_2^{\;2}+y_2^{\;2}}$$

$$=\frac{i(y_2x_2-x_1y_2)}{x_2^{\;2}+y_2^{\;2}}$$

$\frac{z_1}{z_2}$ is purely imaging

Conversely assuming that $\frac{z_1}{z_2}$ is purely imaging and we shall prove that

$|z_1+z_2|^2=|z_1|^2+|z_2|^2$

$\because\ \ \frac{z_1}{z_2}=\frac{i(y_1x_2-x_1y_2)}{x_2^{\;2}+y_2^{\;2}}$ (purely imaging i.e. Real part 0)

$$\frac{z_1}{z_2}=\frac{x_1x_2+y_1y_2+i(y_1x_2-x_1y_2)}{x_2^{\;2}+y_2^{\;2}}$$

$$=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$$

$$\Rightarrow x_1x_2+y_1y_2=0$$

$$\Rightarrow 2x_1x_2+2y_1y_2=0$$

$$\Rightarrow x_1^{\;2}+x_2^{\;2}+2x_1x_2+y_1^{\;2}+y_2^{\;2}+2y_1y_2=x_1^{\;2}+x_2^{\;2}+y_1^{\;2}+y_2^{\;2}$$

$$\Rightarrow (x_1+x_2)^2+(y_1+y_2)^2=x_1^{\;2}+y_1^{\;2}+x_2^{\;2}+y_2^{\;2}$$

$$\Rightarrow |(x_1+x_2)+i(y_1+y_2)|^2=|(x_1+iy_1)|^2+|(x_2+iy_2)|^2$$

$$\Rightarrow |z_1+z_2|^2=|z_1|^2+|z_2|^2$$

Q.80 A vector $\vec{x}$ satisfies the equation $\vec{x}\times\vec{b}=\vec{c}\times\vec{b};\vec{x}\times\vec{a}=0$. Prove that $\vec{x}=\vec{c}-\frac{(\vec{a}.\vec{c})}{(\vec{a}.\vec{b})}\vec{b}$ provided $\vec{a}$ and $\vec{b}$ are not perpendicular. **(8)**

Ans:

In question condition must be $\bar{x}\times\bar{a}\neq 0$ instead of $\bar{x}\times\bar{a}=0$

$\bar{x}\times\bar{b}=\bar{c}\times\bar{b},\quad \bar{x}\times\bar{a}=0$

$\Rightarrow \bar{a}\times(\bar{x}\times\bar{b})=\bar{a}\times(\bar{c}\times\bar{b})$

$\Rightarrow (\bar{a}.\bar{b})\bar{x}-(\bar{a}.\bar{x})\bar{b}=(\bar{a}.\bar{b})\bar{c}-(\bar{a}.\bar{c})\bar{b}$

$$\Rightarrow \bar{x}.\frac{(\bar{a}.\bar{x})\bar{b}}{\bar{a}.\bar{b}} = \bar{c} - \frac{\bar{a}.\bar{c}}{\bar{a}.\bar{b}}\bar{b}$$

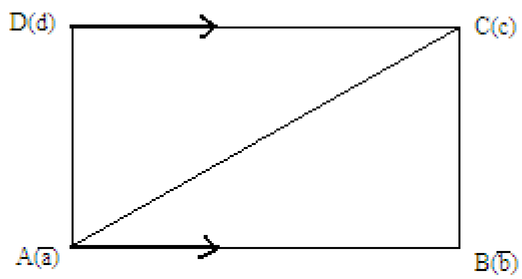
(given condition $\bar{x}\times\bar{a}=0$ is wrong it should be $\bar{x}\times\bar{a}\neq 0$ or $x.a = 0$)

$$\Rightarrow x - \frac{0.b}{a.b} = \bar{c} - \frac{\bar{a}.\bar{c}}{\bar{a}.\bar{b}}$$

$$\Rightarrow x = \bar{c} - \frac{\bar{a}.\bar{c}}{\bar{a}.\bar{b}}\bar{b}$$

Q.81 Using vector methods prove that the diagonals of a parallelogram bisect each other. **(8)**

Ans:



In parallelogram $\overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow$ position vector $(\vec{b}-\vec{a})$ = position vector $(\vec{c}-\vec{d})$

$\Rightarrow \bar{b}+\bar{d} = \bar{c}+\bar{a}$

$\Rightarrow \frac{\bar{b}+\bar{d}}{2} = \frac{\bar{c}+\bar{a}}{2}$

$\Rightarrow$ mid point of $\overrightarrow{BD}$ = mid point of $\overrightarrow{AC}$

$\therefore$ Diagonal of $\|^{gm}$ bisect to each other.

Q.82 The constant forces 2i – 5j + 6k, -i+2j-k and 2i + 7j act on a particle which is displaced from position 4i – 3j – 2k to position 6i + j – 3k. Find the total work done. **(8)**

Ans:

$\vec{F} = \bar{f}_1 + \bar{f}_2 + \bar{f}_3 = 3i + 4\hat{j} + 5k$

Displacement = (6i + j – 3k) - $(4\hat{i} - 3\hat{j} - 2\hat{k})$

$= 2i + 4\hat{j} - \hat{k}$

w = f.d = $(3\hat{i}+4\hat{j}+5k).(2\hat{i}+4\hat{j}-k)$

= 6 + 16 – 5

= 17 N

Q..83 Show that

$$\begin{vmatrix} 1^2+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$ **(8)**

Ans:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Applying $c_1 \rightarrow c_1 - bc_3$ on L.H.S.

$$= \begin{vmatrix} 1+a^2+b^2 & 2ab & -2b \\ 0 & 1-a^2+b^2 & 2a \\ b(1+a^2+b^2) & -2a & 1-a^2-b^2 \end{vmatrix}$$

$R_3 \rightarrow R_3 - bR_1$

$$= \begin{vmatrix} 1+a^2+b^2 & 2ab & -2b \\ 0 & 1-a^2+b^2 & 2a \\ 0 & -2a(1+b^2) & 1-a^2+b^2 \end{vmatrix}$$

$= (1+a^2+b^2)[(1-a^2+b^2)(1-a^2+b^2)+4a^2(1+b^2)]$

$= (1+a^2+b^2)[\{(1+b^2)-a^2\}^2+4a^2(1+b^2)]$

$= (1+a^2+b^2)[(1+b^2)^2+a^4-2a^2(1+b^2)+4a^2(1+b^2)]$

$= (1+a^2+b^2)[(1+b^2)^2+2a^2(1+b^2)+a^4]$

$= (1+a^2+b^2)(1+a^2+b^2)^2$

$= (1+a^2+b^2)^3$ = R.H.S. Hence proved.

Q.84 Write the following equations in the matrix form AX = B and solve for X by finding A^{-1}.

$x + y - 2z = 3$

$2x - y + z = 0$ **(8)**

$3x + y - z = 8$

Ans:

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

$A \times = B$

$$|A| = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} = -5$$

$$a_{11} = (-1)^2 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0, \quad a_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = 5$$

$$a_{13} = (-1)^4 \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 5, \quad a_{21} = (-1)^5 \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1$$

$$a_{22} = (-1)^4 \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 5, \quad a_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = 2, \quad a_{31} = (-1)^4 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -1$$

$$a_{32} = (-1)^5 \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = -5, \quad a_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$Adj.(A) = \begin{bmatrix} 0 & -1 & -1 \\ 5 & 5 & -5 \\ 5 & 2 & 3 \end{bmatrix} \quad A^{-1} = \frac{adjA}{|A|} = -\frac{1}{5}\begin{bmatrix} 0 & -1 & -1 \\ 5 & 5 & -5 \\ 5 & 2 & -3 \end{bmatrix}$$

$$\because A \times = B$$

$$\therefore \times = A^{-1}B = \begin{bmatrix} 0 & 1/5 & 1/5 \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8/5 \\ 5 \\ 9/5 \end{bmatrix} \quad x = 8/5,\ y = 5,\ z = 9/5$$

Q.85 Test the consistency of the following equations and if possible, find the solution

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

$$15x - 3y + 9z = 21$$

(8)

Ans:

Given system of equation

$$\begin{vmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

$$A \times = B$$

$$\text{Now } c = [A : B] = \begin{vmatrix} 4 & -2 & 6 & : 8 \\ 1 & 1 & -3 & : -1 \\ 15 & -3 & 9 & : 21 \end{vmatrix}$$

$$R_1 \Leftrightarrow R_2$$

$$= \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 4 & -2 & 6 & : & 8 \\ 15 & -3 & 9 & : & 21 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_1$$
$$R_3 \to R_3 - 15R_1$$

$$= \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -6 & 18 & : & 12 \\ 0 & -18 & 54 & : & 36 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{6}$$
$$R_3 \to \frac{R_2}{18}$$

$$= \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -1 & 3 & : & 2 \\ 0 & -1 & 3 & : & 2 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$= \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -1 & 3 & : & 2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

R(A) = R(C) < n

$\Rightarrow$ R(A) = R(c) = 2 < 3

$\therefore$ Given system of equation is a consistent

Now we have $\begin{bmatrix} 1 & 1 & -3 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

Let z = k, -y + 3z = 2

-y = 2 – 3k

y = 3k – 2

x + y – 3z = -1

x = -1 – 3k + 2 + 3k

x = +1

Different value of k, system has infinite solution.

Q .86 Obtain the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and use Cayley-Hamilton theorem to find its inverse. **(8)**

Ans:

$|A-\lambda 1|=0$, characteristic equation

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix}=0$$

i.e. $\lambda^3-6\lambda^2+7\lambda+2=0$ ……………..(1)

by using Clayey-Hamilton Theorem, A satisfying (1)

$A^3-6A^2+7A+2I=0$

$\Rightarrow A^2-6A+7I+2A^{-1}=0$

$\Rightarrow 2A^{-1}=-A^2+6A-7I$

$$\Rightarrow 2A^{-1}=-\begin{bmatrix} 5 & 0 & 8 \\ 2 & 1 & 5 \\ 8 & 0 & 13 \end{bmatrix}+6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}-7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 2A^{-1}=\begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ -4 & 0 & -2 \end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{bmatrix}$$

Q.87 Find the Fourier series expansion for the function

$\mathrm{f}(\mathrm{x})=\frac{1}{2}(\pi-\mathrm{x}), 0<\mathrm{x}<2\pi$. **(16)**

Ans:

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ ……………(1)

Now, we have

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)dx$$

$$=\frac{1}{2\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{2\pi}$$

$$a_0=0$$

Now, $a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx dx$

$$=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\cos nx dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \pi \cos nx dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos nx dx$$

$$= 0 - \frac{1}{2\pi}\left[\int_0^{2\pi} \frac{x\sin nx}{n} - \int_0^{2\pi} \frac{\sin nx}{n} dx\right]$$

$$= -\frac{1}{2\pi}\left[\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$a_n = 0$$

Now, $b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} \pi \sin nx dx - \frac{1}{2\pi}\int_0^{2\pi} x\sin nx dx$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} x\sin nx dx$$

$$= -\frac{1}{2\pi}\left[\frac{-x(\cos nx)}{n}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos nx}{n} dx\right]$$

$$= -\frac{1}{2\pi}\left[\frac{-x(\cos nx)}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}\left[-2\pi\frac{\cos 2n\pi}{n}\right]$$

$$= \frac{\cos 2n\pi}{n} = \frac{1}{n}$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

$$\frac{1}{2}(\pi - x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots$$

Q.88 Find the Laplace transform of $L\{t^2 e^t \text{ Sin } 4t\}$. **(8)**

Ans:

$L\{t^2 e^t \sin 4t\}$

$\because$ we know that $L\{\sin 4t\} = \frac{4}{s^2 + 16}$

$\therefore \quad L\{e^t \sin 4t\} = \frac{4}{(s-1)^2 + 16} = f(s)$

$L\{te^t \sin 4t\} = -(1)^1 \frac{d}{ds} f(s)$

$$= -\frac{d}{ds}\left[\frac{4}{s^2-2s+17}\right]$$

$$= \frac{4(2s-2)}{(s^2-2s+17)^2} = F(s)$$

Now, $L\{t^2 e^t \sin 4t\} = -\frac{d}{ds}F(s)$

$$= -\frac{d}{ds}\left[\frac{8(s-1)}{(s^2-2s+17)^2}\right]$$

$$= \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

Q.89 Find the Inverse Laplace transform of $\mathrm{L}^{-1}\left(\frac{s+1}{s^2+6s+25}\right)$ **(8)**

Ans:

$$L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{(s+3)^2+16}\right\}$$

$$= L^{-1}\left\{\frac{s+3-2}{(s+3)^2+16}\right\}$$

$$= L^{-1}\left\{\frac{s+3}{(s+3)^2+16}\right\} + 2L^{-1}\left\{\frac{1}{(s+3)^2+16}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s}{s^2+16}\right\} - 2e^{-3t}L^{-1}\left\{\frac{1}{s^2+16}\right\}$$

$$= e^{-3t}.\cos 4t - 2e^{-3t}.\frac{1}{4}\sin 4t$$

$$= \frac{1}{2}e^{-3t}[2\cos 4t - \sin 4t]$$

Q.90 Solve the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathrm{Sin}2x\ .$$ **(8)**

Ans:

$(D^2-5D+6)y = \sin 2x$

$A.E.\ m^2-5m+6=0$

m = 2, 3

C.F. = $c_1e^{2x} + c_2e^{3x}$

$$\text{P.I.} = \frac{1}{(D^2 - 5D + 6)}\sin 2x$$
$$= \frac{1}{-2^2 - 5D + 6}\sin 2x$$
$$= \frac{1}{-4 - 5D + 6}\sin 2x$$
$$= \frac{1}{-(5D - 2)}\sin 2x$$
$$= \frac{-(5D + 2)}{25d^2 - 4}\sin 2x$$
$$= \frac{-(5D + 2)}{25 \times -4 - 4}\sin 2x$$
$$= \frac{-(5D + 2)}{-104}\sin 2x$$
$$= \frac{10}{104}\cos 2x + 2\sin 2x$$
$$= \frac{5}{52}\cos 2x + \frac{1}{52}\sin 2x$$

y = C.F + P.I

$$= \left(c_1 e^{2x} + c_2 e^{3x}\right) + \frac{5}{52}\cos 2x + \frac{1}{52}\sin 2x$$

Q.91 By using Laplace transform solve the differential equation

$$\frac{d^2 y}{dt^2} + y = t\cos 2t, \text{ with initial conditions } y = 0, \frac{dy}{dt} = 0, \text{ when } t = 0.$$ **(8)**

Ans:

$(D^2 + 1)y = t\cos 2t$ ………………..(1)

$y = 0, \dfrac{dy}{dt} = 0, t = 0$

Taking Laplace transform of equation (1)

$L(y'' + y) = L\{t\cos 2t\}$

$$s^2 L\{y\} - sy(0) - y'(0) + L\{y\} = -\frac{d}{ds}\left\{\frac{s}{s^2 + 4}\right\}$$
$$(s^2 + 1)L\{y\} = -\frac{1}{s^2 + 4} + \frac{2s^2}{s^2 + 4}$$
$$L\{y\} = \frac{s^2 - 4}{(s^2 + 1)(s^2 + 4)^2}$$
$$L\{y\} = -\frac{5}{9}\frac{1}{(s^2 + 1)} + \frac{5}{9}\frac{1}{(s^2 + 4)} + \frac{8}{3}\left\{\frac{1}{(s^2 + 4)^2}\right\}$$

$$y = -\frac{5}{9}L^{-1}\left\{\frac{1}{(s^2+1)}\right\} + \frac{5}{9}L^{-1}\left\{\frac{1}{(s^2+4)}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\int_0^t \frac{1}{2}\sin 2x.\frac{1}{2}\sin 2(t-x)dx$$

By convolution

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\int_0^t \{\cos(2t-4x) - \cos 2t\}dx$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\left[-\frac{1}{4}\sin(2t-4x) - x\cos 2t\right]_0^t$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{12}\sin 2t - \frac{1}{3}t\cos 2t + \frac{1}{12}\sin 2t$$

$$= -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{1}{3}t\cos 2t$$

Q.92 Find the moment of the force $\overline{\text{F}}$ about a line through the origin having direction of $2\hat{i} + 2\hat{j} + \hat{k}$, due to a 30 Kg force acting at a point (– 4, 2, 5) in the direction of $12\hat{i} - 4\hat{j} - 3\hat{k}$. **(8)**

Ans:

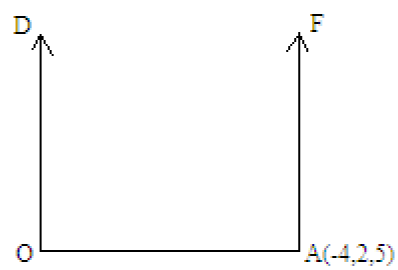
Let D be given line through the origin O and $\vec{F}$ be the force through A(-4, 3, 5).

$\overrightarrow{OA} = -4\hat{i} + 2\hat{j} - 5\hat{k}$

$\vec{F} = \dfrac{30(12\hat{i} - 4\hat{j} - 3\hat{k})}{13}$

$\therefore$ Moment of $\vec{F}$ about $O = \overrightarrow{OA} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 2 & -5 \\ \frac{360}{13} & \frac{-120}{13} & \frac{-90}{13} \end{vmatrix} = -\frac{60}{13}(13\hat{i}$$



Thus the moment of $\vec{f}$ about the line $\vec{D}$

$$-\frac{60}{13}(13\hat{i} + 36\hat{j} + 4\hat{k}).\left(\frac{2i+2j+k}{\sqrt{4+4+1}}\right)$$

$$-\frac{20}{13}(26+72+4) = -\frac{2040}{13} = \frac{2040}{13}$$

Q.93 Prove that the right bisectors of the sides of a triangle intersect at its circum centre. **(8)**

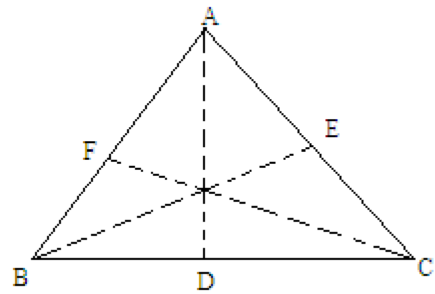
Ans:

Let A,B,C be the vertices of Δ ABC, the mid-point of the sides BC, CA and AB are D,E,F let ⊥ at D and E to BC and CA respectively

interests the point $P(\vec{R})$; then $D\vec{P}.B\overleftarrow{C}=0$

$$\left(\vec{R}-\frac{\vec{B}+\vec{C}}{2}\right).\left(\vec{C}-\vec{B}\right)=0 \qquad \ldots\ldots\ldots.(1)$$

And $E\vec{P}.C\vec{A}=0 \Rightarrow \left(\vec{R}-\frac{\vec{A}+\vec{C}}{2}\right).\left(\vec{A}-\vec{C}\right)=0$ ……(2)



Adding (1) & (2) , we get

$$\left(\overline{R}-\frac{\overline{A}+\overline{B}}{2}\right).\left(\overline{A}-\overline{B}\right)=0 \text{ so } \overline{FP}\perp\overline{AB}$$

$$\Rightarrow \overline{PA}=\overline{PB} \text{ if } \left|\vec{A}-\vec{R}\right|=\left|\vec{B}-\vec{R}\right|$$

$$\Rightarrow \left(\overline{R}-\frac{\overline{A}+\overline{B}}{2}\right).\left(\overline{A}-\overline{B}\right)=0 \qquad \text{Ans.}$$

Q..94 Show that the components of a vector $\overline{B}$ along and perpendicular to $\overline{A}$ in the plane of $\overline{A}$ and $\overline{B}$ are $\left(\frac{\overline{A}\cdot\overline{B}}{A^2}\right)\overline{A}$ and $\frac{\left(\overline{A}\times\overline{B}\right)\times\overline{A}}{A^2}$. **(8)**

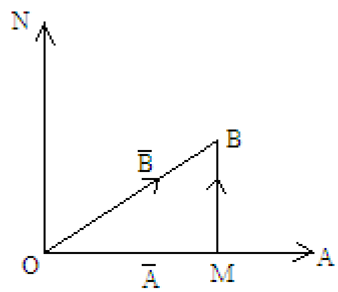
Ans:

Let $O\vec{A}=\vec{A}, O\vec{B}=\vec{B}$ and OM be the projection of $\vec{B}$ on $\vec{A}$.

Component of $\vec{B}$ along $\vec{A}$ = OM

$$= \left(\vec{B}.\hat{A}\right)\hat{A}=\left(\frac{\vec{B}.\vec{A}}{A}\right)\frac{\vec{A}}{A}$$

$$\Rightarrow \frac{\vec{B}.\vec{A}}{A^2}.\vec{A}$$



Also component of $\vec{B}\perp\vec{A}=M\vec{B}=O\vec{B}-O\vec{M}=\vec{B}-\frac{\vec{B}.\vec{A}}{A^2}.\vec{A}$

$$=\frac{(\vec{A}\times\vec{B})\times\vec{A}}{A^2} \qquad \text{Ans.}$$

Q.95 If $\tan(\theta+i\varphi)=e^{i\alpha}$ show that $\theta=\left(n+\frac{1}{2}\right)\frac{\pi}{2}$ and $\phi=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$. **(8)**

Ans:

$$\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\tan 2\theta = \frac{2\cos\alpha}{1-\cos^2\alpha + \sin^2\alpha} = \frac{2\cos\alpha}{0} \to \infty$$

$$\therefore \Rightarrow 2\theta = n\pi + \frac{\pi}{2} \Rightarrow \theta = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$$

Also $\tan 2i\phi = \dfrac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$

$= i\tanh\phi = i\sin\alpha$

Or

$$\Rightarrow \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$ (By Componendo and Devidendo)

$$\Rightarrow \frac{e^{2\phi}}{e^{-2\phi}} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

$$\Rightarrow e^{4\phi} = \left(\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right)^2 \Rightarrow e^{2\phi} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

$$\Rightarrow \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$ Ans.

Q.96 If $(a_1 + ib_1)(a_2 + ib_2)\ldots\ldots\ldots\ldots\ldots\ldots(a_n + ib_n) = A + iB$ then

$$\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots\ldots\ldots\ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}.$$ **(8)**

Ans:

Let $a_j + ib_j = r_j(\cos\alpha_j + i\sin\alpha_j),\ j = 1,2 \_\_\_ n,$

$\Rightarrow A + iB = R(\cos\theta + i\sin\theta)$

Now $(a_1 + ib_1)(a_2 + ib_2) + \ldots\ldots + (a_n + ib_n) = A + iB$

$\Rightarrow r_1 r_2 \ldots\ldots r_n\left[\cos(\alpha_1 + \alpha_2 + \ldots\ldots\alpha_n) + i\sin(\alpha_1 + \alpha_2 \ldots\ldots\alpha_n)\right] = R(\cos\theta + i\sin\theta)$

$\Rightarrow R = r_1, r_2 \ldots\ldots r_n$

Or

$A^2 + B^2 = \left(a_1^{\ 2} + b_1^{\ 2}\right)\left(a_2^{\ 2} + b_2^{\ 2}\right)\ldots\ldots\ldots\left(a_n^{\ 2} + b_n^{\ 2}\right)$

$\Rightarrow \tan^{-1}\dfrac{b_1}{a_1} + \tan\dfrac{b_2}{a_2} + \ldots\ldots\ldots\tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}$

Q.97 Show that the origin and the complex numbers represented by the roots of the equation $z^2 + az + b = 0$, where *a, b* are real, form an equilateral triangle if $a^2 = 3b$. **(8)**

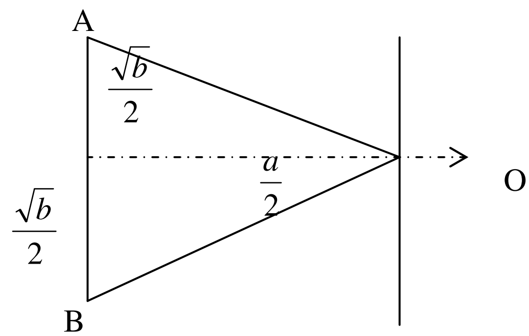
Ans:

$$Z^2 + aZ + b = 0 \Rightarrow Z = \frac{-a \pm i\sqrt{b}}{2}$$

$$\Rightarrow OA = \sqrt{\frac{a^2}{4} + \frac{b}{4}} = \sqrt{b}$$

$$OB = \sqrt{b}$$

Thus $\quad AB = \frac{\sqrt{b}}{2} + \frac{\sqrt{b}}{2} = \sqrt{b}$



$\therefore OA = OB = AB$, hence they form an equilateral triangle.

Q.98 Prove that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$ **(8)**

Ans:

$$\Delta = abcd \begin{vmatrix} a^{-1}+1 & a^{-1} & a^{-1} & a^{-1} \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

$R_1 \to R_1 + (R_2 + R_3 + R_4)$

$$\Rightarrow abcd(1 + a^{-1} + b^{-1} + c^{-1} + d^{-1}) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$

$$= abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ b^{-1} & 1 & 0 & 0 \\ c^{-1} & 0 & 1 & 0 \\ d^{-1} & 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \qquad \text{Ans.}$$

Q.99 Determine the values of α, β, γ when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal. **(8)**

Ans:

If A is orthogonal then $AA' = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} \Rightarrow 4\beta^2 + \gamma^2 &= 1 \\ 2\beta^2 - \gamma^2 &= 0 \end{aligned}\right\} \ldots\ldots \quad \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}$$

But $\alpha^2 + \beta^2 + \gamma^2 = 1 \Rightarrow \alpha \pm \frac{1}{\sqrt{2}}$

$$\Rightarrow \alpha \pm \frac{1}{\sqrt{2}}, \beta \pm \frac{1}{\sqrt{6}}, \gamma \pm \frac{1}{\sqrt{3}} \quad \text{Ans.}$$

Q.100 Find the values of k such that the system of equations $x + ky + 3z = 0$, $4x + 3y + kz = 0$, $2x + y + 2z = 0$ has non-trivial solution. **(8)**

Ans:

The set of equation is $\begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$R_2 \to R_2 - 4R_1, R_3 \to R_3 - 2R_1$$

For a non-trivial solution ρ(A) = ρ(A : B) = 2

Thus $|A| = 0$

$$A = \begin{bmatrix} 1 & k & 3 \\ 0 & 3-4k & k-12 \\ 0 & 1-2k & -4 \end{bmatrix}$$

$\Rightarrow$ -4(3 – 4k) – (1 – 2k)(k – 12) = 0

$$\Rightarrow 2k^2 - 9k = 0 \Rightarrow k = 0, \frac{9}{2}$$

Q.101 Find the characteristic equation of the matrix $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$. Hence find A^{-1}.

(8)

Ans:

Characteristic equation is $\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$ or $A^3 - 6A^2 + 6A - 11I = 0$

$\Rightarrow A^{-1} = \frac{1}{11}\left[A^2 - 6A + 6I\right]$

$$\Rightarrow A^{-1} = \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Q.102 Find the Fourier series for $f(t) = \begin{cases} 0, & -2 < t < -1 \\ 1+t, & -1 < t < 0 \\ 1-t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$. **(16)**

Ans:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos\frac{n\pi t}{2} + \sum_{n=1}^{\infty} bn \sin\frac{n\pi t}{2}$$

$$a_0 = \frac{1}{2}\left[\int_{-2}^{-1} 0dt + \int_{-1}^{0}(1+t)dt + \int_{0}^{1}(1-t)dt + \int_{1}^{2} 0dt\right] = \frac{1}{2}$$

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0}(1+t)\cos\frac{n\pi t}{2}dt + \int (1-t)\cos\frac{n\pi t}{2}dt\right]$$

$$= \frac{1}{2}\left[\left\{(1+t)\sin\frac{n\pi t}{2}.\frac{2}{n\pi} + \frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right\}_{-1}^{0} + \left\{(1-t)\sin\frac{n\pi t}{2}.\frac{2}{n\pi} - \frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right\}_{0}^{1}\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{2}\left[\int_{-1}^{0}(1+t)\sin\frac{n\pi t}{2}dt + \int_{0}^{1}(1-t)\sin\frac{n\pi t}{2}dt\right]$$

$$= \frac{1}{2}\left[\left\{-(1+t)\cos\frac{n\pi t}{2}.\frac{2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi t}{2}\right\}_{-1}^{0} + \left\{(1-t) - \left(\cos\frac{n\pi t}{2}.\frac{2}{n\pi}\right) - \frac{4}{n^2\pi^2}\sin\frac{n\pi t}{2}\right\}_{0}^{1}\right]$$

$$b_n = 0$$

$$\Rightarrow \therefore f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi}{2}t \quad \text{Ans.}$$

Q.103 Find $L\left(e^{-4t}\ \frac{sin\,3t}{t}\right)$. **(8)**

Ans:

$$L\{\sin 3t\} = \frac{3}{s^2+9}, \quad L\left\{\frac{\sin 3t}{t}\right\} = \int_s^\infty \frac{3}{s^2+9}$$

$$\Rightarrow \left\{\tan^{-1}\frac{s}{3}\right\}_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{3}$$

$$\Rightarrow L\left\{\frac{\sin 3t}{t}\right\} = \cot^{-1}\frac{s}{3}$$

$$\Rightarrow L\left\{e^{-4t}\frac{\sin 3t}{t}\right\} = \cot^{-1}\frac{s+4}{3}$$

$$\Rightarrow \tan^{-1}\frac{3}{s+4} \qquad \text{Ans.}$$

Q.104 Find the inverse Laplace transform of $\frac{s+4}{s(s-1)\left(s^2+4\right)}$. **(8)**

Ans:

$$\frac{s+4}{s(s-1)(s^2+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow A=-1, \quad B=1, \quad C=0, \quad D=-1$$

$$\Rightarrow L^{-1}\left\{\frac{s+4}{s(s-1)(s^2+4)}\right\} = L^{-1}\left\{-\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}\right\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\Rightarrow -1 + e^{-t} - \frac{1}{2}\sin 2t \qquad \text{Ans.}$$

Q..105 Using Laplace transformation, solve the following differential equation:

$\frac{d^2x}{dt^2} + 9x = \cos 2t$ if $x(0) = 1$, $x\left(\pi/2\right) = -1$. **(8)**

Ans:

$$L\left\{\frac{d^2x}{dt^2}\right\} + 9L\{x\} = L\{\cos 2t\}$$

$$\Rightarrow s^2\overline{X}(s) - sX(0) - X'(0) + 9\overline{X}(s) = \frac{s}{s^2+4}$$

$$\Rightarrow \left(s^2+9\right)\overline{X}(s) - X'(0) = \frac{s}{s^2+4} + s = \frac{s^3+5s}{s^2+4}$$

$$\Rightarrow \overline{X}(s) = \frac{s(s^2+5)}{(s^2+4)(s^2+9)} + \frac{X'(0)}{(s^2+9)}$$

Taking Laplace Inverse transform

$$X(t) = \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{4}{5}L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{X'(0)}{s^2+9}\right\}$$

$$X(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{1}{3}X'(0)\sin 3t$$

Put $t = \frac{\pi}{t}$ we get $x\left(\frac{\pi}{t}\right) = 1$

$$-1 = -\frac{1}{5} + \frac{1}{3}X'(0)$$

$$X'(0) = \frac{-12}{5}$$

$$X = \frac{1}{5}[\cos 2t + 4\cos 3t - 12\sin 3t]$$

Q..106 If z is any complex number and $\bar{z}$ is its complex conjugate then show that $z\bar{z} = |z|^2$. **(7)**

Ans:

Let $z = x + iy$ then $\bar{z} = x - iy$

Now $z\bar{z} = (x+iy)(x - iy) = x^2 + y^2 ----(1)$

Also $|z|^2 = \left|\sqrt{x^2+y^2}\right|^2 = (x^2 + y^2)$----------(2)

From (1) and (2), $z\bar{z} = |z|^2$

Q..107 Find the square root of the complex number 3 + 4i. **(7)**

Ans:

Let $\sqrt{3+4i} = \pm(x+iy)$, Then $3+4i = (x+iy)^2 = x^2 - y^2 + 2i\,xy$

$\Rightarrow x^2 - y^2 = 3$ ------------(1) and $xy = 2$ -------------(2)

Now, $(x^2+y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 9+16 = 25$

$\Rightarrow x^2 + y^2 = 5$ ------------(3)

from (1) and (3) $x^2=4$, $y^2=1$ $\Rightarrow x = \pm 2, y = \pm 1$

from (2) xy is positive so if x=2, y=1 and x=-2, y=-1

Hence $\sqrt{3+4i} = \pm(2+i)$

Q..108 If $z = \text{Cos}\,\theta + i\,\text{Sin}\,\theta$ then find $z^n + \frac{1}{z^n}$. **(7)**

Ans:

Given $z = \text{Cos}\,\theta + i\,\text{Sin}\,\theta \Rightarrow z^n = \text{Cos}\,n\,\theta + i\,\text{Sin}\,n\,\theta$, $z^{-n} = \text{Cos}n\theta - i\text{Sin}n\theta$ Therefore $z^n + z^{-n} = 2\,\text{Cos}\,n\,\theta$.

Q..109 If $a_r = \text{Cos}\left(\pi/2^r\right) + i\,\text{Sin}\left(\pi/2^r\right)$ $r = 1,2,3,....$ then show that $a_1\,a_2\,a_3...\text{ad inf} = -1$.

(7)

Ans:

Given $a_r = \text{Cos}\left(\dfrac{\pi}{2^r}\right) + i\,\text{Sin}\left(\dfrac{\pi}{2^r}\right) = e^{i\frac{\pi}{2^r}}$

$\Rightarrow a_1 = e^{i\frac{\pi}{2^r}}, a_2 = e^{i\frac{\pi}{2^2}}, ...$

Now

$a_1 a_2 a_3................$

$= e^{i\frac{\pi}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+--------\right)} = e^{i\pi}$ = Cos π + i Sin π = -1

Q..110. If a square matrix A is invertible then show that A^T (transpose of A) is also invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$. **(7)**

Ans:

Since A is invertible matrix, therefore $|A| \neq 0 \Rightarrow |A^T| \neq 0$

$\Rightarrow A^T$ is also invertible

Now $AA^{-1} = I = A^{-1}A \Rightarrow (AA^{-1})^T = I = (A^{-1}A)^T \Rightarrow (A^{-1})^T A^T = I = A^T(A^{-1})^T$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

Q..111 Compute the inverse of the matrix $A = \begin{pmatrix} 3 & -4 & 2 \\ 0 & 5 & 9 \\ -4 & 8 & 1 \end{pmatrix}$. **(7)**

Ans:

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 5 & 9 \\ -4 & 8 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -4 & 2 \\ 0 & 5 & 9 \\ -4 & 8 & 1 \end{vmatrix} = -17 \neq 0$$

$$Adj\ A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \text{ and } C_{ij} = (-1)^{i+j} \text{ minor of } C_{ij}$$

$$C_{11} = \begin{vmatrix} 5 & 9 \\ 8 & 1 \end{vmatrix} = 5 - 72 = -67,\ C_{12} = -\begin{vmatrix} 0 & 9 \\ -4 & 1 \end{vmatrix} = -36$$

$$C_{13} = \begin{vmatrix} 0 & 5 \\ -4 & 8 \end{vmatrix} = 20,\ C_{21} = -\begin{vmatrix} -4 & 2 \\ 8 & 1 \end{vmatrix} = 20$$

$$C_{22} = \begin{vmatrix} 3 & 2 \\ -4 & 1 \end{vmatrix} = 11\ , C_{23} = -\begin{vmatrix} 3 & -4 \\ -4 & 8 \end{vmatrix} = -8$$

$$C_{31} = \begin{vmatrix} -4 & 2 \\ 5 & 9 \end{vmatrix} = -46\ , C_{32} = -\begin{vmatrix} 3 & 2 \\ 0 & 9 \end{vmatrix} = -27\ , C_{33} = \begin{vmatrix} 3 & -4 \\ 0 & 5 \end{vmatrix} = 15$$

$$\text{Adj } A = \begin{bmatrix} -67 & 20 & -46 \\ -36 & 11 & -27 \\ 20 & -8 & 15 \end{bmatrix}$$

$$\text{Now } A^{-1} = \frac{1}{|A|}\text{Adj } A$$

$$= -\frac{1}{17}\begin{bmatrix} -67 & 20 & -46 \\ -36 & 11 & -27 \\ 20 & -8 & 15 \end{bmatrix}$$

Q..112. Evaluate $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix}$ where ω is a complex cube root of unity. **(7)**

Ans:

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} = \left[1-\omega^3\right] - \omega\left[\omega^2 - \omega^2\right] + \omega^2\left[\omega^4 - \omega\right] = 0, \text{ Since } \omega^3 = 1$$

Q..113 Show without evaluating that determinant $\begin{vmatrix} 1 & x & y+z \\ 1 & y & x+z \\ 1 & z & x+y \end{vmatrix} = 0$. **(7)**

Ans:

$$\begin{vmatrix} 1 & x & y+z \\ 1 & y & x+z \\ 1 & z & x+y \end{vmatrix}$$

$C_2 \rightarrow C_2+C_3$

$$= \begin{vmatrix} 1 & x+y+z & y+z \\ 1 & x+y+z & x+z \\ 1 & x+y+z & x+y \end{vmatrix}$$

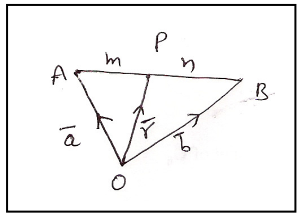
$$= (x+y+z)\begin{vmatrix} 1 & 1 & y+z \\ 1 & 1 & x+z \\ 1 & 1 & x+y \end{vmatrix} = (x+y+z)\ 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

$= 0$

Q..114 Find the position vector of a point which divides the line joining two given points in three dimensional space. **(7)**

Ans:

Let the position vectors of points A and B are $\bar{a}$ and $\bar{b}$ respectively. Let P be the point which divides the line joining A and B in the ratio m:n and let $\bar{r}$ be the position vector of P. Then $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OP} = \bar{r}$ where O is origin



Given $\frac{AP}{PB} = \frac{m}{n} \Rightarrow AP = \frac{m}{n} PB$

$\because \overline{AP}$ and $\overline{PB}$ are collinear

$\therefore \overline{AP} = \frac{m}{n}\overline{PB}$ or $n\,\overline{AP} = m\overline{PB}$ – – – – – – – – (1)

Now $\overline{AP} = \overline{OP} - \overline{OA} = \bar{r} - \bar{a}$, $\overline{PB} = \overline{OB} - \overline{OP} = \bar{b} - \bar{r}$

From (i) we get $n\,\bar{r} + m\bar{r} = n\bar{a} + m\bar{b}$

$$\bar{r} = \frac{n\bar{a} + m\bar{b}}{n+m}$$

Q..115. Show that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ and $3\vec{i} - 4\vec{j} - 4\vec{k}$ form the sides of a right angled triangle. **(7)**

Ans:

Let $\bar{A} = 2i - j + k$, $\bar{B} = i - 3j - 5k$ $\bar{C} = 3i - 4j - 4k$

$\bar{A}.\bar{B} = 2 + 3 - 5 = 0$, $|\bar{A}| = \sqrt{6}, |\bar{B}| = \sqrt{35}, |\bar{C}| = \sqrt{41},$

$\Rightarrow$ sides represented by $\bar{A}$ and $\bar{B}$ are at right angles

Also $|\bar{C}|^2 = |\bar{A}|^2 + |\bar{B}|^2$

$\therefore$ vectors $\bar{A}, \bar{B}$ and $\bar{C}$ form the sides of right angled triangle

Q..116. State Cayley Hamilton Theorem and verify it for the square matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$. **(7)**

Ans:

<u>Cayley Hamilton Theorem</u>

Every square matrix satisfies its own characteristic equation

$$\text{Let } A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Characteristic matrix is

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Characteristic equation is

$|A-\lambda I| = 0 \Rightarrow \lambda^3-7\lambda^2+11\lambda-5 = 0$. By Cayley Hamilton theorem

$A^3-7A^2+11A-5I = 0$

$$\text{Now} \quad A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 32 & 62 & 31 \\ 31 & 63 & 31 \\ 31 & 62 & 32 \end{bmatrix}$$

$A^3-7A^2+11A-5I$

$$= \begin{bmatrix} 32 & 62 & 31 \\ 31 & 63 & 31 \\ 31 & 62 & 32 \end{bmatrix} - \begin{bmatrix} 49 & 84 & 42 \\ 42 & 91 & 42 \\ 42 & 84 & 49 \end{bmatrix}$$

$$+ \begin{bmatrix} 22 & 22 & 11 \\ 11 & 33 & 11 \\ 11 & 22 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q..117 Show that the system of equations

$2x - 3y + z = 0$

$x + 2y - 3z = 0$

$4x - y - 2z = 0$

has only the trivial solution. **(7)**

Ans:

System of equations is $2x - 3y + z = 0$, $x + 2y - 3z = 0$ and $4x - y - 2z = 0$

This is system of homogeneous equations can be written as

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or AX = O, where

$$A=\begin{bmatrix}2 & -3 & 1\\ 1 & 2 & -3\\ 4 & -1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, O=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now $|A| = \begin{vmatrix}2 & -3 & 1\\ 1 & 2 & -3\\ 4 & -1 & -2\end{vmatrix} = 7 \neq 0$

Thus $|A| \neq 0$, So, the given system has only the trivial solution given by x=y=z=0

Q..118 Find the Fourier Series for the function,

$f(x) = x,\ 0 < x < 2\pi.$ **(14)**

Ans:

The Fourier series of $f(x)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{ Cosnx} + \sum_{n=1}^{\infty} b_n \text{ Sin nx}$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \text{ Cos nx dx}$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \text{ Sin nx dx}$

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} xdx = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \text{ Cos nx } dx = \frac{1}{\pi}\left[x\frac{\text{Sin n x}}{\text{n}} - \left(-\frac{\text{Cos n x}}{\text{n}^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{1}{\text{n}^2} - \frac{1}{\text{n}^2}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \text{x Sin nx dx} = \frac{1}{\pi}\left[-\text{x}\frac{\text{Cos n x}}{\text{n}} + \frac{\text{Sin n x}}{\text{n}^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi}{\text{n}}\right] = -\frac{2}{\text{n}}$$

Fourier series is

$$x = \pi - 2\sum_{n=1}^{\infty}\frac{Sin \text{ n x}}{\text{n}}$$

Q..119 Distinguish between even and odd functions. Give one example for each of these functions. **(7)**

Ans:

Even function:

A function $f(x)$ is said to be even function if $f(-x) = f(x)$

Odd Function:

A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$

Example:

Cos x, x^2 are even functions and Sin x, x are odd functions

Q..120. Forces $2\vec{i}+7\vec{j},\ 2\vec{i}-5\vec{j}+6\vec{k},\ -\vec{i}+2\vec{j}-\vec{k}$ act on a point P having position vector $4\vec{i}-3\vec{j}-2\vec{k}$. Find the vector moment of the resultant of three forces acting at P about the point Q whose position vector is $6\vec{i}+\vec{j}-3\vec{k}$. **(7)**

Ans:

Let $\bar{R}$ be the resultant of the three forces, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3$ = 3i + 4j + 5k

Vector moment of $\bar{R}$ at P about Q

$= \overline{PQ} \times \bar{R}$

=(2i+4j-k) × (3i+4j+5k)

$$\begin{vmatrix} i & j & k \\ 2 & 4 & -1 \\ 3 & 4 & 5 \end{vmatrix} = i\,[20+4]-j\,[10+3]+k\,[8-12] = 24i-13j-4k$$

Q..121 Define Laplace transform of a function. Obtain the Laplace transform of Cosh (at). **(7)**

Ans:

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$, where function f(t) is defined for t ≥ 0 and s>0 is a parameter.

$$L[\text{Cos h}(at)] = \int_0^\infty e^{-st}\text{Cos h}(at)\,dt$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t}+e^{-(s+a)t}\right]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}+\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right] = \frac{1}{2}\frac{2s}{s^2-a^2} = \frac{s}{s^2-a^2}$$

Q..122 Find the inverse Laplace transform of $\frac{s-1}{s^2-6s+25}$. **(7)**

Ans:

$$L^{-1}\left[\frac{s-1}{s^2-6s+25}\right] = L^{-1}\left[\frac{s-1}{(s-3)^2+16}\right]$$

$$= L^{-1}\left[\frac{s-3}{(s-3)^2+16}\right] + 2L^{-1}\left[\frac{1}{(s-3)^2+16}\right]$$

$$= e^{3t}\,L^{-1}\left[\frac{s}{s^2+16}\right] + 2e^{3t}L^{-1}\left[\frac{1}{s^2+16}\right] = e^{3t}\,\text{Cos}\,4t + 2e^{3t}\,\frac{1}{4}\,Sin\,4t$$

$$= e^{3t}\left[Cos\,4t + \frac{1}{2}\text{Sin}\,4t\right]$$

Q..123 Solve the differential equation $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$. **(7)**

Ans:

Differential equation is $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

Let $y = e^{mx}$ is the solution of given differential equation.

The auxiliary equation is $m^2-7m+12 = 0 \Rightarrow (m-4)(m-3) = 0 \Rightarrow m=3,4$

solution is $y=C_1e^{3x} + C_2e^{4x}$

Q..124 Solve by using Laplace transform, the differential equation

$$\frac{d^2y}{dt^2} + 4y = \sin t,\ y(0) = 1,\ y'(0) = 0. \quad \textbf{(7)}$$

Ans:

Given $\frac{d^2y}{dt^2} + 4y = Sin\ t$

Taking Laplace transform of both sides, we have

$$L\left[\frac{d^2y}{dt^2}\right] + 4\,L[y] = L[\text{Sin t}] \Rightarrow s^2\,L[y] - sy(0) - y'(0) + 4L[y] = \frac{1}{s^2+1}$$

But $y(0) = 1,\ y'(0) = 0 \Rightarrow (s^2+4)\,L[y] - s = \frac{1}{s^2+1}$

$$L[y] = \frac{s}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)}$$

$$y = L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right] = Cos\ 2t + \frac{1}{3}\text{Sin t} - \frac{1}{6}\text{Sin 2t}$$