TYPICAL QUESTIONS & ANSWERS

PART I

OBJECTIVE TYPES QUESTIONS

Each Question carries 2 marks.

Choose correct or the best alternative in the following:

Q.1 Which of the following is an entire function

(A) $\frac{z}{1+z^2}$ **(B)** $z\bar{z}$

(C) e^{-z^2} **(D)** $e^{Z^{-2}}$

Ans: C

Q.2 Let $f(z)=|z|^3$. Then which of the following statements is not correct.

(A) f is differentiable at z = 0. **(B)** f is differentiable at $z \neq 0$.

(C) f is not analytic at z = 0. **(D)** f is not analytic at $z \neq 0$.

Ans: C

Q.3 The image of a square under the transformation $w = \sqrt{2}e^{\pi i/7}z+1$ is

(A) a square **(B)** a circle

(C) the upper half plane **(D)** the right half plane

Ans: A

Q.4 The value of $\int_{|z|=2}\left(\frac{e^z}{z}+\sin z\right)dz$ is

(A) $2\pi i$ **(B)** 0

(C) $4\pi i$ **(D)** $\frac{e^2}{2}+\sin 2$

Ans: A

Q.5 The value of line integral $\int_C \left(xy^2\vec{i}+y^3\vec{j}\right).d\vec{r}$, where C is the segment of the parabola $y^2 = x$ from (0, 0) to (1, 1) is

(A) $\frac{12}{7}$ **(B)** 0

(C) $\frac{7}{12}$ **(D)** $\frac{1}{3}$

Ans: C

Q.6 The directional derivative of $xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$ is

(A) $\frac{-11}{3}$ **(B)** $\frac{-3}{11}$

(C) 0 **(D)** 1

Ans: A

Q.7 The unit normal to the surface $x^2 + y^2 + z^2 = 1$ at the point $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ is

(A) $\frac{1}{\sqrt{2}}(\vec{i} - \vec{j})$ **(B)** $\frac{1}{\sqrt{2}}(-\vec{i} + \vec{j})$

(C) $\frac{1}{\sqrt{2}}(\vec{i} + \vec{j})$ **(D)** None of the above

Ans: C

Q.8 Which of the following probability mass functions can define a probability distribution

(A) $f(x) = \frac{5 - x^2}{6}$ for $x = 0,1,2,3$ **(B)** $f(x) = \frac{x}{15}$ for $x = 1,2,3,4,5$

(C) $f(x) = \frac{1}{2^x}$ for $x = 0,1,2,3,4$ **(D)** $f(x) = \frac{1}{4}$ for $x = 2,3,4,5,6$

Ans: B

Q.9 The expected value of a random variable X is 3 and its variance is 2. Then the variance of 2X + 5 is

(A) 8 **(B)** 9

(C) 10 **(D)** 11

Ans: A

Q.10 The equation $y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} - \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$ is

(A) Elliptic **(B)** Parabolic

(C) Hyperbolic **(D)** None of the above

Ans: B

Q.11 The image of the line $\mathrm{Im}(z) = 1$ under the mapping $\omega = z^2$ is a

(A) Circle. **(B)** Parabola.

(C) Hyperbola. **(D)** Ellipse.

Ans: B

Q.12 The values of z for which e^z is real is

(A) any multiple of π. **(B)** odd multiples of $\pi/2$.

(C) all multiples of $\pi/2$. **(D)** any real number.

Ans: A

Q.13 The value of $\lim_{z\to 2i}\left(3x+iy^2\right)$ is

(A) 2 **(B)** 0
(C) - 4i **(D)** i

Ans: C

Q.14 Let $f(z)=\bar{z}$. Then which of the following is true:

(A) $f(z)$ is not continuous at z = 0
(B) $f(z)$ is differentiable at z = 0
(C) $f(z)$ is not continuous at z = 0 but differentiable at z = 0
(D) $f(z)$ is continuous at z = 0 but not differentiable at z = 0

Ans: D

Q.15 The value of integral $\int_{1}^{2-i} z\,dz$ is

(A) 0 **(B)** 1
(C) 1+2i **(D)** $1-2i$

Ans: D

Q.16 The angle between the surfaces $x\log z = y^2-1$ and $x^2y = 2-z$ at the point (1, 1, 1) is

(A) $\cos^{-1}\frac{1}{\sqrt{2}}$ **(B)** $\cos^{-1}\frac{1}{\sqrt{30}}$

(C) $\cos^{-1}\frac{1}{\sqrt{15}}$ **(D)** $\cos^{-1}\frac{1}{\sqrt{10}}$

Ans: B

Q.17 A force field $\bar{F}$ is said to be conservative if

(A) Curl $\bar{F}=0$ **(B)** grad $\bar{F}=0$
(C) Div $\bar{F}=0$ **(D)** Curl (grad $\bar{F}$) = 0

Ans: A

Q.18 The partial differential equation $x^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}=0$ is an example of

(A) Hyperbolic equation **(B)** Elliptic equation
(C) Parabolic equation **(D)** None of these

Ans: C

Q.19 The inequality between mean and variance of Binomial distribution which is true is

(A) Mean < Variance **(B)** Mean = Variance
(C) Mean > Variance **(D)** Mean × Variance = 1

Ans: C

Q.20 Let f (x) be a probability density function defined by $f(x) = e^{-x}$, for $x \geq 0$ and f (x) = 0 for x< 0, then the value of cumulative distribution function at x =2 is

(A) $1 + e^{-2}$ **(B)** $1 - e^{-2}$
(C) $1 + e^{2}$ **(D)** $1 + e^{-2.5}$

Ans: B

Q.21 If $w = u(x, y) + i\, v(x, y)$ is an analytic function of $z = x + i\, y$, then $\frac{dw}{dz}$ equals

(A) $i\frac{\partial w}{\partial x}$ **(B)** $-i\frac{\partial w}{\partial x}$

(C) $i\frac{\partial w}{\partial y}$ **(D)** $-i\frac{\partial w}{\partial y}$

Ans: D

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] = -i\frac{\partial w}{\partial y}$$

Q.22 The image of the circle $|z| = 2$ under the mapping w = z +(3+2 i) is a

(A) circle. **(B)** ellipse.
(C) pair of lines. **(D)** hyperbola.

Ans: A

$$\because z = w - (3 + 2i)$$
$$|z| = |u - 3 + i(v - 2)|$$
$$\Rightarrow (u-3)^2 + (v-2)^2 = 4$$

which is circle in (u,v) plane with centre (3,2).

Q.23 The function $(z-1)\sin\frac{1}{z}$ at $z = 0$ has

(A) a removable singularity.
(B) a simple pole.
(C) an essential singularity.
(D) a multiple pole.

Ans: C

Let $f(z) = (z-1)\sin\frac{1}{z}$ $= (z-1)\left[z^{-1} - \frac{z^{-3}}{\lfloor 3} + \frac{z^{-5}}{\lfloor 5} - \text{-------}\right]$

$$= \left(1 - \frac{1}{z} - \frac{1}{\lfloor 3\, z^2} + \frac{1}{\lfloor 3\, z^3} - \text{------}\right)$$

Z = 0 is an essential singularity.

Q.24 A unit normal vector to the surface $x^3 - x\,y\,z = 1$, at the point (1, 1, 1) is

(A) $\frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$.

(B) $\frac{2\,\hat{i} - \hat{j} + \hat{k}}{\sqrt{6}}$.

(C) $\frac{2\,\hat{i} - \hat{j} + \hat{k}}{\sqrt{6}}$.

(D) $\frac{2\,\hat{i} - \hat{j}}{\sqrt{5}}$.

Ans: B

$$\frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$$

Let $\phi = x^3$-xyz-1,

The normal to surface at point (1,1,1) is

$$\bar{\bar{\nabla}}\phi = (3x^2 - yz)\hat{i} - xz\hat{j} - xy\hat{k}$$

$$\bar{\bar{\nabla}}\phi_{(1,1,1)} = 2\hat{i} - \hat{j} - \hat{k}$$

$$\hat{n} = \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$$

Q.25 The line integral $\int_c x^2 dx + y^2 dy$, where C is the boundary of the region $x^2 + y^2 \le a^2$ equals

(A) 0.

(B) a.

(C) πa^2

(D) $\frac{1}{2}\pi a^2$

Ans: A

$\int_C (x^2 dx + y^2 dy) = \iint_R \left(\frac{\partial}{\partial y} x^2 - \frac{\partial}{\partial x} y^2\right) dx dy = 0$ Using Green's Lemma.

Q.26 The partial differential equation $y\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0$ is elliptic if

(A) $x^2 = y^2$

(B) $x^2 < y^2$

(C) $x^2 + y^2 > 1$

(D) $x^2 + y^2 = 1$

Ans: B

The given differential equation is

$$y\frac{\partial^2 u}{\partial x^2}+2x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=0$$

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}=0$$

It is elliptic if B^2-4AC<0 i.e. $x^2<y^2$

Q.27 The expected value of a random variable X is 2 and its variance is 1, then variance of 3X+4 is

(A) 9. **(B)** 7.
(C) 3. **(D)** 13.

Ans: A

$V(3x+4)=E((3x+4)^2) - (E(3x+4))^2$

$= 9E(x^2)+24E(x)+16-(3E(x)+4)^2 = 9E(x^2)-9(E(x))^2 = 9V(x)=9$

Q.28 Let X be a random variable having a normal distribution. If P (X < 0) = P(X > 2) = 0.4, then mean value of X equals

(A) 0. **(B)** 1.
(C) 1.5 **(D)** 2.

Ans: B

As P(X < 0) = P(X > 2). $\Rightarrow$ 0, 2 are symmetrically placed around μ.

$$\Rightarrow \mu = \frac{0+2}{2}=1.$$

Q.29 The image of the point z = 2+3*i* under the transformation w = z(z-2*i*) is

(A) $1+2i$ **(B)** $1+8i$
(C) 0 **(D)** 1-8*i*

Ans.: **B**

By definition, if a point z_0 maps into the point w_0 through the transformation w = f(z) then w_0 is known as the image of z_0. Consequently,

w = f(2+3*i*) = (2+3i)(2+3i-2i) = (2+3*i*)(2+*i*)=1+8*i*

Q.30 Let (i) and (ii) denote the facts

(i) : f is continuous at z = 0
(ii) : f is differentiable at z = 0

Then for function $f(z)=\overline{z}$ which is correct statement?

(A) both (i) & (ii) are true **(B)** (i) is true, (ii) is false
(C) (i) is false, (ii) is true **(D)** both (i) & (ii) are false.

Ans. **B**

$f(z)=\overline{z}=\lim_{x\to 0, y\to 0}(x-iy)=0=f(0)$ f is continuous at z = 0.

$$f'(0) = \underset{\Delta z \to 0}{Lt} \frac{\Delta \bar{z}}{\Delta z} = \underset{\Delta z \to 0}{Lt} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{Lt} \left(\frac{1 - i\frac{\Delta y}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}} \right)$$

$$= \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{Lt} \left(\frac{1-m}{1+m} \right) \text{ which depends on slope m.}$$

Therefore, f is not differentiable at z = 0.

Q.31 The order of the pole of the function $f(z) = \frac{z^4 + 2z + 1}{z^2 + 5z + 2}$, at $z = \infty$ is

(A) 2 **(B)** 1
(C) 0 **(D)** 4

Ans. **A**

$z = \frac{1}{w}, f\left(\frac{1}{w}\right) = \frac{1}{w^2}\left[\frac{1 + 2w^3 + w^4}{1 + 5w + 2w^2}\right]$ w=0, is a pole of order 2.

Q.32 The value of the integral $\oint_C \frac{3z^2 + 7z + 1}{z - 3} dz$ where C is circle $|z| = 2$ traversed clockwise, is

(A) 2π **(B)** $6\pi i$
(C) 0 **(D)** 1

Ans. **C**

z=3 lies outside the circle. By Cauchy's integral theorem.

Q.33 The curl of the gradient of a scalar function U is

(A) 1 (B) $\nabla^2 U$
(C) ∇U (D) 0

Ans. **D**

$$\text{Curl girdU} = \nabla \wedge \left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k} \right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z \partial y} \right) + \hat{j}\left(\frac{\partial^2 u}{\partial z \partial x} - \frac{\partial^2 u}{\partial x \partial z} \right) + \hat{k}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} \right)$$

= 0.

Q.34 The value of the integral $\int_C y dS$ where C is the curve $y = 2\sqrt{x}$ from x = 3 to x = 24 is

(A) 156 **(B)** 153
(C) 150 **(D)** 158

Ans. **A** $\int_C y dS = \int_3^{24} 2\sqrt{x}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_3^{24} 2\sqrt{x+1} dx = 156$

Q.35 The tangent vector to the curve whose parametric equation is $x = t^3, y = \frac{t+1}{t}, z = t^2 + 1$ at t=2 is given by

(A) $12\hat{i} + \frac{1}{4}\hat{j} - 4\hat{k}$ **(B)** $12\hat{i} - \frac{1}{4}\hat{j} + 4\hat{k}$

(C) $12\hat{i} - \frac{1}{4}\hat{j} - 4\hat{k}$ **(D)** $12\hat{i} + \frac{1}{4}\hat{j} + 4\hat{k}$

Ans. **B**

The tangent vector to any curve is given by

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\hat{i}x + \hat{j}y + \hat{k}z\right) = \frac{d}{dt}\left(\hat{i}t^3 + \hat{j}\left(\frac{t+1}{t}\right) + \hat{k}\left(t^2+1\right)\right)$$

$$= \left[\hat{i}\left(3t^2\right) + \hat{j}\left(-\frac{1}{t^2}\right) + \hat{k}(2t)\right]_{t=2} = 12\hat{i} - \frac{1}{4}\hat{j} + 4\hat{k}$$

Q.36. The cumulative distribution function F of a continuous variate X is such that F(a)=0.5, F(b)=0.7. Then value of P(a≤X≤b) is given as

(A) 0 **(B)** 0.5

(C) 0.2 **(D)** 0.7

Ans. **C** $P(a \le X \le b) = F(b) - F(a) = 0.2$

Q.37 A discrete random variate X has probability mass function f which is positive at x = -1,0,1 and is zero elsewhere. If f(0)=0.5 then the value of $E(X^2)$ is

(A) 1 **(B)** 0

(C) 0.5 **(D)** -0.5

Ans. **C**

$$E(X^2) = (-1)^2 f(-1) + 0f(0) + f(1) = f(1) + f(-1) = 1 - f(0) = 0.5$$

Q.38 A room has three lamp sockets. From a collection of 10 light bulbs of which only 6 are good,
a person selects 3 at random and puts them in a socket. What is the probability that the room will have light?

(A) 29/120 **(B)** 39/60

(C) 19/30 **(D)** 29/30

Ans. Answer is not given. Misprint in the question paper.

$$p = \frac{3}{5}, q = \frac{2}{5}. P(X \ge 1)P(X = 1) + P(X = 2) + P(X = 3) = \sum_{n=1}^{3} {}^3C_n \left(\frac{6}{10}\right)^n \left(\frac{4}{10}\right)^{3-n} = \frac{936}{1000}$$

Q.39 If the ends x = 0 and x = L are insulated in one dimensional heat flow problems, then the boundary conditions are

(A) $\frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(L,t)}{\partial x} = 1$ at t =0 **(B)** $\frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(L,t)}{\partial x} = 0$ at t = 0.

(C) $\frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(L,t)}{\partial x} = 0$ for all t **(D)** $\frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(L,t)}{\partial x} = 1$ for all t

Ans.: **C** As end points are insulated so by definition.

Q.40 If $u = x^2 + t^2$ is a solution of $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ then the value of c is

(A) 1 **(B)** 2
(C) -2 **(D)** -1/2

Ans. **A** Using partial derivative, we get $\because c^2 = 1 \Rightarrow c = 1$.c= ± 1, the negative sign is neglected on physical grounds

Q.41 The curves u(x, y) = C and v(x, y) = C' are orthogonal if

(A)u and v are complex functions **(B)**u+*i*v is an analytic function
(C)u – v is analytic function **(D)**u + v is an analytic function

Ans. **B** By definition of orthogonal function.

Q.42 The value of $\int_0^{1+i} z^2 dz$ along the line x =y is

(A) 2/3 **(B)** -2/3
(C) -2*i*/3 **(D)** 2*i*/3

Ans. Answer is not given. Correct solution is $\int_0^{1+i} z^2 dz = \int_0^1 (1+i)^3 x^2 dx = \frac{2i}{3}(1+i)$

Q.43 The critical points of transformation $w = z + \frac{1}{z}$ are given as

(A) ±1 **(B)** ±*i*
(C) ±*i*/2 **(D)** ±1/2

Ans. **B** By definition.

Q.44 If the mean and variance of binomial variate are 12 and 4, then the probabilities of the distribution are given by the terms in the expansion of

(A) $\left(\frac{1}{3} + \frac{2}{3}\right)^{12}$ **(B)** $\left(\frac{1}{3} + \frac{2}{3}\right)^{4}$

(C) $\left(\frac{1}{3} + \frac{2}{3}\right)^{18}$ **(D)** $\left(\frac{1}{2} + \frac{1}{2}\right)^{18}$

Ans. **C** Mean= np=12, variance = npq=4$\Rightarrow p = 2/3, q = 1/3, n = 18$

Q.45 The E(e^{tx}), t<0.5 of a random variable X is $(1\text{-}2t)^{-4}$, then E(X) is given as

(A) 5 **(B)** 6
(C) 7 **(D)** 8

Ans. **D** As E'(e^{tx}) = $8(1\text{-}2t)^{-5}$ Therefore E(X) is obtained for t = 0, thus E(X)=8

Q.46 If $\bar{F}$ is a conservative force field, then the value of curl $\bar{F}$ is

(A) 0 **(B)** 1
(C) $\bar{\nabla}\bar{F}$ **(D)** -1

Ans. **A** By definition of a conservative field $\vec{F} = \nabla\varphi$ and curl $\nabla\varphi = 0$.

Q.47 If $u = x^2 + y^2 + z^2$, $\vec{V} = xi + yj + zk$, then value of $\nabla.\left(u\vec{V}\right)$ is equal to

(A) 5u **(B)** $5\left|\bar{V}\right|$
(C) $5(u+\left|\bar{V}\right|)$ **(D)** $5(u-\left|\bar{V}\right|)$

Ans. **A**

Use the identity div $\left(u\vec{V}\right)$=udiv$\vec{V}$ + $\vec{V}$ .gradu

$= \left(x^2+y^2+z^2\right)3 + \left(\hat{i}x+\hat{j}y+\hat{k}z\right).\left(\hat{i}2x+\hat{j}2y+\hat{k}2z\right)$

$= 3x^2+3y^2+3z^2+2x^2+2y^2+2z^2 = 5\left(x^2+y^2+z^2\right) = 5u$

Q.48 If $\nabla.\bar{F} = 3$ then $\iint_S \bar{F}.\hat{N}dS$, where S is the surface of unit sphere is

(A) 3π **(B)** 5π
(C) 4π **(D)** 6π

Ans. **C**

$$\iint_S \mathrm{F.\hat{N}dS} = \iiint_S \mathrm{divFdV} = 3\iiint_{V(\text{unit sphere})} \mathrm{dxdydz} = 3\int_0^1\int_0^{\pi}\int_0^{2\pi} \mathrm{r^2dr.\sin\theta d\phi d\varphi}$$

$$= 3.\left(\frac{r^3}{3}\right)_0^1 \left(-\cos\theta\right)_0^{\pi} \left(\varphi\right)_0^{2\pi} = 2.2\pi = 4\pi\,.$$

Q.49 The partial differential equation of a vibration of a string is

(A) $\frac{\partial^2 y}{\partial t^2} = \frac{\partial y}{\partial x}$ **(B)** $\frac{\partial^2 y}{\partial t^2} = \frac{\partial y}{\partial t}$
(C) $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ **(D)** $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial y}{\partial x}$

Ans.: **C**

It is not partial differential equation but an ordinary differential equation as y is a function of 't' only. The partial differential equation governing the vibration of a string is a wave equation which is hyperbolic.

Q.50 The partial differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as

(A) parabolic **(B)** elliptic
(C) hyperbolic **(D)** none of these

Ans. **B**

For pde $Af_{xx} + 2Bf_{xy} + Cf_{yy} = 0$ to be classified as parabolic, eclipse or hyperbolic $B^2 - AC \lesseqgtr 0$. Here this discriminant is <0. Hence elliptic.

Q.51 The value of $\int_C \frac{e^z}{(z-3)^2} dz, C : |z| = 2$ is

(A) 3 **(B)** 2
(C) 1 **(D)** 0

Ans. **D**

Singularities lie outside $|z| = 2$. Therefore by Cauchy's integral theorem D follows.

Q.52 Residue of tanz at z = π/2 is

(A) -1 **(B)** 1
(C) 0 **(D)** 2

Ans. **A** $\operatorname{Re} sf\left(\frac{\pi}{2}\right) = \underset{z\to\frac{\pi}{2}}{lt} \frac{sinz}{\frac{d}{dz}(\cos z)} = -1$

Q.53 The Taylor series expansion of $\frac{1}{z-2}, |z| < 1$ is

(A) $-\frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} ....\right)$ **(B)** $-\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} ....\right)$

(C) $-\frac{1}{2}\left(1 - \frac{z}{2} - \frac{z^2}{4} + \frac{z^3}{8} ....\right)$ **(D)** $-\frac{1}{2}\left(1 + \frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{8} ....\right)$

Ans. **B** $\frac{1}{z-2} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} ....\right)$

Q.54 Let X be normal with mean 10 and variance 4, then P(X<11) is

(A) $\phi\left(1/2\right)$ **(B)** $\phi\left(-1/2\right)$

(C) $\phi(1)$ **(D)** $\phi(-1)$

Ans. **A** $\phi(z) = P\left(\frac{X-10}{2}\right) \Rightarrow P(X < 11) = \phi\left(1/2\right)$

Q.55 If $f(x) = \frac{K}{1+x^2}, -\infty < x < \infty$ is a valid probability mass function of x, then the value of K is

(A) π **(B)** $1/\pi$

(C) 1/2 **(D)** 2

Ans. **B** $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{K}{1+x^2} dx = 1 \Rightarrow 2k \tan^{-1} x\Big|_0^{\infty} = 1 \Rightarrow k = \frac{1}{\pi}$

Q.56 If X is random variable representing the outcome of the roll of an ideal die, then E(X) is

(A) 3 **(B)** 2.5
(C) 3.5 **(D)** 4

Ans. **C** $E(X)=\sum_{n=1}^{6} n\frac{1}{6}=\frac{7}{2}=3.5$

Q.57 If S is a closed surface enclosing a volume V and if $\overline{R}=x\hat{i}+y\hat{j}+z\hat{k}$ then $\iint_S \overline{R}\Box\overline{N}dS$ is equal to

(A) $3\,\overline{R}$ **(B)** 3S
(C) 3V **(D)** $3\,\overline{N}$

Ans. **C** $\iint_S \overline{R}\Box\overline{N}dS=\int_V div\overline{R}dV=3\int_V dV=3V$

Q.58 The unit normal at (2,-2,3) to the surface $x^2y+2xy=4$ is

(A) $\frac{1}{3}\left(\hat{i}-2\hat{j}+2\hat{k}\right)$ **(B)** $\frac{1}{3}\left(\hat{i}-2\hat{j}-2\hat{k}\right)$
(C) $\frac{1}{3}\left(-\hat{i}-2\hat{j}+2\hat{k}\right)$ **(D)** $\frac{1}{3}\left(-\hat{i}+2\hat{j}+2\hat{k}\right)$

Ans. Answer is not given. Misprint in Question.

$$\hat{n}=\frac{\nabla f}{|\nabla f|}=\frac{\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}}{\sqrt{f_x^2+f_y^2+f_z^2}}=\frac{(2yx+2y)\hat{i}+\left(2x+x^2\right)\hat{j}}{\sqrt{(2yx+2y)^2+\left(2x+x^2\right)^2}}$$

At the point (2, -2, 3) we get $\frac{-12\hat{i}+8\hat{j}}{\sqrt{144+64}}$

Q.59 Eliminating a and b from the $(x-a)^2+(y-b)^2+z^2=c^2$, we obtain the partial differential equation

(A) $z^2(p-q+1)=c^2$ **(B)** $z^2(p^2+q^2+1)=c^2$
(C) $z^2(p^2+q^2)=c^2$ **(D)** $z^2(p-q)=c^2$

Ans.: **B** $zp=-(x-a),\ zq=-(y-b)\Rightarrow z^2(p^2+q^2+1)=c^2$

Q.60 Solution of $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$ is

(A) $\sqrt{x}-\sqrt{y}=f\left(\sqrt{x}-\sqrt{z}\right)$ **(B)** $\sqrt{x}+\sqrt{y}=f\left(\sqrt{x}-\sqrt{z}\right)$
(C) $\sqrt{x}-\sqrt{y}=f\left(\sqrt{x}+\sqrt{z}\right)$ **(D)** $\sqrt{x}+\sqrt{y}=f\left(\sqrt{x}+\sqrt{z}\right)$

Ans. **A** Using Lagrange's subsidiary equations we get

$$\because \frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}\Rightarrow\sqrt{x}-\sqrt{y}=f\left(\sqrt{x}-\sqrt{z}\right).$$

Q.61 Residue of $\frac{\cos z}{z}$ at z = 0 is

(A) 1 **(B)** -1
(C) 2 **(D)** 0

Ans. **A** $\frac{\cos z}{z} = \frac{1}{z}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ....\right]$ Residue is the coefficient of $\frac{1}{z}$ .i.e. 1

Q.62 The function w = log z is analytic everywhere except at the value of z when z is equal to

(A) -1 **(B)** 1
(C) 2 **(D)** 0

Ans. **D** $w = \log z \Rightarrow \frac{dw}{dz} = \frac{1}{z}, z \neq 0$

Q.63 If f(z) is analytic in a simply connected domain D and c is any simple closed curve inside D, then the value of $\int_c f(z)dz$ is given by

(A) 1 **(B)** 2
(C) 0 **(D)** 3

Ans. **C** By Cauchy's integral theorem.

Q.64 If X is a binomial variate with p = 1/5, for the experiment of 50 trials, then the standard
deviation is equal to

(A) 6 **(B)** -8
(C) 8 **(D)** $2\sqrt{2}$

Ans. **D** As variance = npq = 50(0.2)(0.8)=8 $\Rightarrow$ S.D. = square root of variance = $\sqrt{8}$ = $2\sqrt{2}$

Q.65 A unit normal to $x^2 + y^2 + z^2 = 5$ at (0,1,2) is equal to

(A) $\frac{1}{\sqrt{5}}(i + j + k)$ **(B)** $\frac{1}{\sqrt{5}}(i + j - k)$
(C) $\frac{1}{\sqrt{5}}(j + 2k)$ **(D)** $\frac{1}{\sqrt{5}}(i - j + k)$

Ans. **C**

$\nabla(x^2 + y^2 + z^2) = (2xi + 2yj + 2zk)_{(0,1,2)} = 2j + 4k. \Rightarrow \hat{n} = \frac{2j + 4k}{\sqrt{4+16}}$

Q.66 If $u = x^2 + y^2 + z^2,\ \vec{V} = xi + yj + zk,$ then value of $\nabla.\left(u\vec{V}\right)$ is equal to

(A) 2u **(B)** -u
(C) 3u **(D)** 5u

Ans. **D**

Using the identity $div(u\vec{V}) = u div\vec{V} + \vec{V}.grad u$

$= \left(x^2 + y^2 + z^2\right)3 + \left(\hat{i}x + \hat{j}y + \hat{k}z\right)\left(2\hat{i}x + 2\hat{j}y + 2\hat{k}z\right)$

$= 5\left(x^2 + y^2 + z^2\right) = 5u$.

Q.67 $\int_c \overline{F}.d\overline{R}$ is independent of the path joining any two points if it is

(A) irrotational field **(B)** solenoidal field
(C) rotational field **(D)** vector field

Ans. **A**

Since the integral is independent of path therefore the vector
$\vec{F} = \nabla\varphi \rightarrow curl\nabla\varphi = 0$ which further implies that the field is irrotational.

Q.68 The solution of the partial differential equation $\frac{\partial^2 z}{\partial y^2} = x^4\sin(xy)$ is

(A) $z = -x^2\sin(xy) + yf(x) + g(x)$
(B) $z = -x^2 sm(xy) - xf(x) + g(x)$
(C) $z = -y^2\sin(xy) + yf(x) + g(x)$
(D) $z = x^2\sin(xy) + yf(x) + g(x)$

Ans. **A**

Q.69 When a vibrating string has an initial velocity, its initial conditions are

(A) $\left[\frac{\partial y}{\partial t}\right]_{t=0} = 0$ **(B)** $\left[\frac{\partial y}{\partial t}\right]_{t=0} = v$

(C) $\left[\frac{\partial y}{\partial t}\right]_{t=0} = \infty$ **(D)** None of these

Ans. **B**

Q.70 Image of $|z+1| = 1$ under the mapping w = 1/z is

(A) 2 Im w + 1 = 0 **(B)** 2 Im w - 1 = 0
(C) 2 RI w + 1 = 0 **(D)** 2 RI w - 1 = 0

Ans. **C**

Q.71 The value of $\int_c \frac{4z^2 + z + 5}{z - 4}dz$ where *C:* is $9x^2 + 4y^2 = 36$ equal to

(A) -1 **(B)** 1
(C) 2 **(D)** 0

Ans. **D**

Q.72 The invariant points of the transformation w=(l + z)/(l—z) are given by

(A) $\pm I$ **(B)** ±2
(C) 0 **(D)** ±1

Ans. A

Q.73 In a Poisson Distribution if 2P (x = 1) = P (x=2), then the variance is

(A) 4 **(B)** 2
(C) 3 **(D)** 1

Ans. A

Q.74 If V(X)=2, then V(2X+3) is equal to

(A) 6 **(B)** -8
(C) 8 **(D)** $2\sqrt{2}$

Ans. C

Q.75 $div(curl\,\overline{F})$ is equal to

(A) 0 **(B)** -1
(C) $\frac{1}{\sqrt{5}}\left(\hat{j}+2\hat{k}\right)$ **(D)** $\frac{1}{\sqrt{5}}\left(\hat{i}-\hat{j}+\hat{k}\right)$

Ans. A

Q.76 *If* $\phi = 3x^2y-y^3z^2$, *grad* ϕ *at (1,-2,-1) is* equal to

(A) $-(12\hat{i}+9\hat{j}+16\hat{k})$ **(B)** $(12\hat{i}+5\hat{j}+8\hat{k})$
(C) $-(12\hat{i}-5\hat{j}+8\hat{k})$ **(D)** $-(12\hat{i}+5\hat{j}-8\hat{k})$

Ans. A

Q.77 If $\overline{A}$ is such that $\nabla \times \overline{A} = 0$ then $\overline{A}$ is called

(A) irrotational **(B)** solenoidal
(C) rotational **(D)** none of these

Ans. A

Q.78 If $f(x) = kx^3, 0 < x < 1$ and 0 elsewhere, is a p.d.f. then the value of k is equal to

(A) 4 **(B)** 2
(C) 3 **(D)** 1

Ans. A $\int_0^1 kx^3 dx = 1 \Rightarrow \left|kx^4\right|_0^1 = 4 \Rightarrow k = 4$

Q.79 Let $f(z) = z\overline{z}$. Then which of the following statements is not correct

(A) f (z) is differentiable at z = 0.
(B) f (z) is differentiable at $z \neq 0$.

(C) f (z) is not analytic at z = 0.
(D) f (z) is not analytic at any point $z \neq 0$.

Ans: **B**

Q.80 Which of the following mapping is conformal at z = 0

(A) $w = z^2$. **(B)** $w = \frac{1}{z}$.
(C) w = cos z. **(D)** w = sin z.

Ans: **D**

Q.81 The Taylor's series about z = 2 of the function $f(z) = \frac{(z+3)}{(z-1)(z-4)}$ converges in the region

(A) $|z| < 1$. **(B)** $|z| < 4$.
(C) $|z-2| < 1$. **(D)** $|z-1| < 3$.

Ans: **C**

Q.82 If $v = u + f(u)$, then grad v equals

(A) $(1 + f'(u))\,\text{grad}\, u$. **(B)** $f'(u)\,\text{grad}\, u$.
(C) $f'(u)$ + grad u. **(D)** grad u.

Ans: **A**

Q.83 The surface integral $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ equals

(A) 0. **(B)** $\frac{4}{3}\pi$.

(C) 2π. **(D)** $\frac{8}{3}\pi$.

Ans: **D**

Q.84 From an urn containing 4 white, 5 black and 6 blue balls, 5 balls are chosen at random with replacement. The expected number of blue balls selected is

(A) 2.5. **(B)** 2.
(C) 1.5. **(D)** 1.

Ans: **B**

Q.85 The mean and variance of a binomial probability distribution are 1 and $\frac{2}{3}$ respectively, then the probability that random variable takes value 0 is

(A) $\frac{8}{27}$. **(B)** $\frac{6}{27}$.
(C) $\frac{3}{27}$. **(D)** $\frac{1}{27}$.

Ans: **A**

Q.86 One dimensional heat equation is given by

(A) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. **(B)** $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.
(C) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. **(D)** $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$.

Ans: **B**

PART – II

NUMERICALS

Q.1 Compute the limit $\lim_{z\to 0}\left[\left(1-e^{z}\right)^{-1}\int_0^z \frac{d\xi}{1+\xi^2}\right]$. **(8)**

Ans:

By Taylor series expansion,

$(1+\xi^2)^{-1} = 1-\xi^2+\xi^4-\xi^6+......$

Therefore,

$$\int_0^2 (1+\xi^2)^{-1}d\xi = z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^6}{6}+......$$

Similarly, the Taylor series expansion of $1 - e^z$ gives,

$$\frac{1}{1-e^z} = -\frac{1}{z}\left(1+\frac{z}{2}+\frac{z^2}{3!}+...\right)^{-1}$$

$$= -\frac{1}{z}\left(1-\frac{z}{2}-\frac{z^2}{3!}-...\right)$$

$$= \left(-\frac{1}{z}+\frac{1}{2}+\frac{z}{3!}----\right)$$

Therefore,

$$\left[(1-e^z)^{-1}\int_0^z \frac{d\xi}{1+\xi^2}\right] = -1+O(z).$$

Hence, $$\lim_{z\to 0}\left[(1-e^z)^{-1}\int_0^z \frac{d\xi}{1+\xi^2}\right] = -1.$$

Q.2 Show that $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n+\frac{1}{z^n}\right)$, where

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos n\theta \cosh(2\cos\theta)d\theta.$$ **(8)**

Ans:

By the Taylor series expansion,

$$\cosh\left(z+\frac{1}{z}\right) = 1+\frac{1}{2!}\left(z+\frac{1}{z}\right)^2+\frac{1}{4!}\left(z+\frac{1}{z}\right)^4+....$$

$$= a_0+\sum_{n=1}^{\infty}a_n\left(z^n+\frac{1}{z^n}\right)$$

for some $a_0, ..., a_n, ...$

From the Laurent series expansion of $\cosh\left(z+\frac{1}{z}\right)$ we have,

$$\cosh\left(z+\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{2\pi i}\int_c \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}dz, \ C : |z| = r, \text{for all } r > 0$$

Let r = 1, $z = e^{i\theta}$ then

$$a_n = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \cosh(e^{i\theta}+e^{-i\theta})e^{-in\theta} i d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)(\cos n\theta - i\sin n\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)\cos n\theta d\theta - \frac{i}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)\sin n\theta d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta)\cos n\theta d\theta$$

as the second integral is 0

Q.3 Evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x} dx$, by contour integration. **(10)**

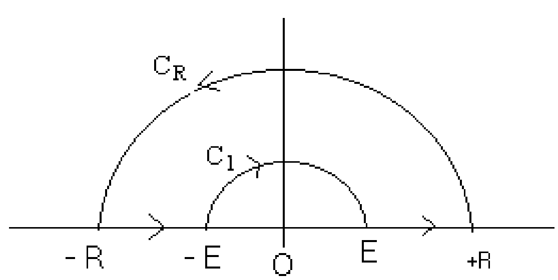
Ans:

$$\frac{\sin^3 x}{x^3} = \frac{1}{4}\text{Im}\left(\frac{3e^{ix}-e^{3ix}}{x^3}\right)$$

Consider the integral

$$\int_c f(z)dz = \frac{1}{4}\int_c \left(\frac{3e^{ix}-e^{3ix}}{x^3}\right)$$

where C is the contour given in figure below



$$\int_C f(z)dz = \int_{C_n} f(z)dz + \int_{-R}^{\in} f(x)dx + \int_{C_1} f(z)dz + \int_{\in}^{R} f(x)dx$$

$$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} \left|\frac{3e^{iz} - e^{3iz}}{z^3}\right| dz$$

$$\le \frac{3e^{-y} + e^{-3y}}{R^3}\pi R$$

$$\to 0 \text{ as } R \to \infty \ (y \to \infty)$$

Since z = 0 is a pole of f (z), by Cauchy-Residue theorem,

$$\int_{C_1} f(z)dz = \int_{C_1} \frac{3e^{iz} - e^{3iz}}{z^3} dz = -\pi i[\operatorname{Res}_{z} = \text{o} f(z)].$$

Now, $\operatorname{Res}_{z} = \text{o} f(z) = \frac{3}{4}$. Therefore, $\int_{C_1} f(z)dz = \frac{-3\pi i}{4}$.

Since f (z) is analytic in C, we have $\int_{C_R} f(z)dz = 0$. Therefore

$$0 = \int_{C_R} f(z)dz + \int_{-R}^{\epsilon} f(x)dx + \int_{C_1} f(z)dz + \int_{\epsilon}^{R} f(x)dx$$

Taking limit R → ∞ and ε → 0 we get

$$0 = 0 + \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx - \frac{3\pi i}{4}$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)dx = \frac{3\pi i}{4}.$$

Hence

$$\operatorname{Im}\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{4}.$$

Q.4 Evaluate $\int_C (z-1)^2 e^{\frac{1}{z^2}} dz$, where C is the circle $\{z : |z| = 2\}$ traversed counter clockwise. **(6)**

Ans:

Let $f(z) = (z-1)^2 e^{\frac{1}{z^2}}$, then $z = 0$ is an essential singularity for f(z). Therefore, by Cauchy - Residue theorem,

$$\int_C (z-1)^2 e^{\frac{1}{x^2}} dz = 2\pi i[\operatorname{Res}_{z=0} f(z)]$$

Now by Taylor series expansion,

$$e^{\frac{1}{z^2}} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{6z^6} + \ldots$$

Therefore,

$$(z-1)^2 e^{\frac{1}{z^2}} = (z^2 - 2z + 1)\left(1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{6z^6} + \ldots\right)$$

$$= z^2 - \frac{2}{z} + 2 + O\left(\frac{1}{z^2}\right)$$

Therefore Residue of f (z) at z = 0 is -2.

Hence

$$\int_C f(z)dz = -4\pi i$$

Q.5 Show that the function $v(r,\theta) = 3r^2 \sin 2\theta - 2r \sin\theta$ in polar coordinates (r, θ) is harmonic. Find the corresponding harmonic conjugate function and construct the analytic function f (z) = u + iv such that f(1) = 1. **(8)**

Ans:

By Cauchy-Riemann equations

$$v_\theta = ru_r \quad v_r = -\frac{1}{r}u_\theta$$

$$\text{i.e., } u_r = \frac{1}{r}v_\theta \quad u_\theta = -rv_r$$

From the definitions $v(r, \theta)$

$$v_\theta = 6r^2 \cos 2\theta - 2r\cos\theta$$

$$v_{\theta\theta} = -12r^2 \sin 2\theta + 2r \sin\theta$$

$$v_r = 6r \sin 2\theta - 2\sin\theta$$

$$v_{rr} = 6\sin 2\theta$$

Therefore,

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 6\sin 2\theta + \frac{1}{r}[6r\sin 2\theta - 2\sin\theta] + \left(-12\sin 2\theta + \frac{2}{r}\sin\theta\right)$$

$$= \sin 2\theta(6 + 6 - 12) + \sin\theta\left(-\frac{2}{r} + \frac{2}{r}\right)$$

$$= 0.$$

i.e., $v(r,\theta)$ is harmonic.

$$u_r = \frac{1}{r}v_\theta = 6r\cos 2\theta - 2\cos 2\theta$$

Therefore, $u(r,\theta) = 3r^2 \cos 2\theta - 2r\cos\theta + \phi(\theta)$

$$u_\theta = -6r^2 \sin 2\theta + 2r\sin\theta + \phi'(\theta)$$

$$v_r = 6r\sin 2\theta - 2\sin\theta$$

$v_r = -\frac{1}{r}u_\theta$ implies $\phi'(\theta) = 0$ and hence $\phi(\theta) = C$, constant.

Therefore, $u(r,\theta) = 3r^2 \cos 2\theta - 2r\cos\theta + C.$

Now,
$f(z) = u + iv$
$= 3r^2 \cos 2\theta - 2r\cos\theta + i(3r^2 \sin 2\theta - 2r\sin\theta) + C$
let $z = re^{i\theta}$, $z^2 = r^2 e^{i2\theta}$
Therefore, $f9z) = 3z^2 - 2z + C$
Since $f(1) = 1$, we get $C = 0$.
hence $f(z) = 3z^2 - 2z.$

Q.6 Find the Fractional Linear Transformation which maps the unit disc $\{z : |z| \le 1\}$ onto the right half plane $\mathrm{Re}(w) \ge 0$. **(8)**

Ans:
Take the points $z_1 = -1, z_2 = i, z_3 = 1$ in the z - plane
and $w_1 = 0, w_2 = i, w_3 = \infty$ on the w - plane.
Then the Cross Ratio
$$\frac{w - w_1}{w - w_3}\frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}$$
implies
$$\frac{w - 0}{w - \infty}\frac{i - \infty}{i - 0} = \frac{z + 1}{z - 1}\frac{i - 1}{i + 1}$$
Now using the convention $\frac{\infty}{\infty} = 1$ we get, $w = -\frac{z+1}{z-1}$
Now it is easy to see that $z = 0$ goes to $w = 1$.
So $|z| \le 1$ maps onto the right half plane under the above transformation.

Q.7 A thin rectangular homogenous thermally conducting plate lies in the xy-plane defined by $0 \le x \le 1,\ 0 \le y \le 1$. The edge y = 0 is held at the temperature x (x –), while the remaining edges are held at temperature 0°. The other faces are insulated and no internal sources and sinks are present. Find the steady state temperature inside the plate. **(10)**

Ans:
The given problem can be formulated as
$\nabla^2 u = 0$
$u(0, y) = u(1, y) = 0$
$u(x,1) = 0,\ u(x,0) = x(x - 1).$

Assume the solution be in the form u (x,y) = X (x)Y (y). Then substituting this in the equation we get,
$X''Y + Y''X = 0$

From this, we get

$$\frac{X''}{X} = \frac{-Y''}{Y} = k \text{ (say)}$$

<u>Case 1:</u> $k > 0$.

Let $k = p^2 > 0$. Then the solution u (x, y) is

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py),$$

Now,

$$u(0, y) = 0 \Rightarrow (c_1 + c_2)(c_3 \cos py + c_4 \sin py) = 0.$$

since $\cos py \neq 0$ and $\sin py \neq 0$, we get $c_1 + c_2 = 0$.

$$u(1, y) = 0 \Rightarrow c_1 e^{\alpha p} + c_2 e^{-\alpha p} = 0.$$

The above relations immediately imply that $c_1 = c_2 = 0$.

This leads trivial solution.

<u>Case 2:</u> $k = 0$

In this case we get

$$u(x, y) = (c_5 x + c_6)(c_7 y + c_8)$$

$$u(0, y) = u(1, y) = 0 \Rightarrow c_5 = c_6 = 0.$$

This again leads to trivial solution.

<u>Case 3:</u> $k < 0$

Let $k = -p^2$. Then

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

$$u(0, y) = 0 \Rightarrow c_1 (c_3 e^{py} + c_4 e^{-py}) = 0$$

since $c_3 e^{py} + c_4 e^{-py} \neq 0$, we get $c_1 = 0$.

$$u(1, y) = 0 \Rightarrow c_2 \sin p (c_3 e^{py} + c_4 e^{-py}) = 0$$

For nontrivial solution $c_2 \neq 0$. Therefore $p = n\pi$. Then

$$u(x, y) = c_2 \sin n\pi x (c_3 e^{n\pi y} + c_4 e^{-n\pi y})$$

By superposition principle,

$$u(x, y) = \sum_{n=1}^{\infty} \sin n\pi x (c_n e^{n\pi y} + d_n e^{-n\pi y})$$

The boundary condition

$$u(x, 1) = 0 \Rightarrow \sin n\pi x (c_n e^{n\pi} + d_n e^{-n\pi}) = 0.$$

This implies $d_n = -c_n e^{2n\pi}$.

Therefore,

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2c_n}{e^{-n\pi}} \sin(n\pi x) \sinh[n\pi(y - 1)]$$

From the boundary condition u (x, 0) = x (x – 1) we get

$$x(x-1) = \sum_{n=1}^{\infty} \frac{2c_n}{e^{-n\pi}} \sin n\pi x \sinh(-n\pi).$$

Which immediately implies

$$\frac{2c_n}{e^{-n\pi}} \sinh(-n\pi) = 2\int_0^1 x(x-1)\sin n\pi x dx$$

$$= \frac{4}{n^3\pi^3}[(-1)^n - 1]$$

Therefore

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh[n\pi(y-1)]$$

where

$$a_n = \frac{e^{-n\pi}}{\sin h(-n\pi)} \frac{4}{n^3\pi^3}[(-1)^n - 1]$$

Q.8 It has been claimed that in 60% of all solar heat installations, the utility bill is reduced by at least one third. Accordingly what are the probabilities that the utility bill will be reduced by at least one-third in

(i) four of five installations,

(ii) at least four of five installations. **(6)**

Ans:

Given problem can be formulated as

$u_t = ku_{xx}, 0 \le x \le 1$

$u(0,t) = u(l,t) = 0$

$u(x,0) = x$

Let u (x, t) = X (x) T (t) be the solution. Then from the equation we get

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \quad \text{(say)}$$

i.e. $X'' + \lambda X = 0, T' + \lambda kT = 0$ The B.C's $u(0,t) = u(l,t) = 0$ implies X(0) = X(l) = 0

The acceptable solution of the Sturm-Liouville problem

$X'' + \lambda X = 0, \quad X(0) = X(l) = 0$

is $\lambda_n = \frac{n^2\pi^2}{\ell^2}$ and $X_n(x) = b_n \sin\frac{n\pi x}{\ell}$.

The solutions of $T' + \frac{n^2\pi^2}{\ell^2} kT = 0$ is $T(t) = c_1 e^{-\frac{n^2\pi^2}{\ell^2}kt}$.

Hence the solution u (x, t) may be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\ell} e^{-\frac{n^2\pi^2}{\ell^2}kt}$$

From the initial condition u (x, 0) = x we get

$$x = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\ell}.$$

Multiplying $\sin\frac{n\pi x}{\ell}$ on both sides and integrating from 0 to ℓ, we get

$$c_n = \frac{2}{\ell}\int_0^1 x\sin\frac{n\pi x}{\ell}dx = \frac{\ell^2}{n\pi}(-1)^{n+1}.$$

Q.9 Let the probability density function of a random variable x be given by $f(x) = \begin{cases} \cos x, & 0 < x < \pi/2, \\ 0, & \text{elsewhere.} \end{cases}$

Find the corresponding cumulative distribution function and determine the probabilities that the random variable x will take a value

(i) greater than $\pi/3$ (ii) between 0 and $\pi/4$. **(8)**

Ans:

The probability density of the random variable is given by

$$f(x) = \begin{cases} \cos x & 0 < x < \pi/2 \\ 0 & \text{elsewhere} \end{cases}$$

So, the density function F (x) is F (x) = $\int_{-\infty}^{x} f(t)dt.$

If $x \le 0$, then $F(x) = \int_{-\infty}^{x} 0dt = 0.$

If $0 < x < \pi/2$, then

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0dt + \int_0^x \cos t dt = \sin t\Big|_0^x = \sin x.$$

If $x > \pi/2$, then

$$F(x) = \int_{-\infty}^{0} 0dt + \int_0^{\pi/2} \cos t dt + \int_{\pi/2}^{x} 0dt = 1.$$

Hence, $F(x) = \begin{cases} 0 & x \le 0 \\ \sin x & 0 < x < \pi/2 \\ 1 & x \ge \pi/2. \end{cases}$

(i) The probability that the random variable will take a value greater than $\pi/3$ is $\int_{\pi/3}^{\infty} f(x)dx = \int_{\pi/3}^{\pi/2} \cos x dx = \sin(\pi/2) - \sin(\pi/3) = 1 - \sqrt{3}/2$

(ii) The probability that the random variable will take a value between 0 and $\pi/4$ is $\int_0^{\pi/4} f(x)dx = \int_0^{\pi/4} \cos x dx = \sin\pi/4 - 0 = 1/\sqrt{2}.$

Q.10 Consider the heat flow in a thin rod of length $\ell, 0 \le x \le \ell$. The ends x = 0 and x = ℓ are insulated. The rod was initially at temperature f (x) = x. By the method of Separation of Variables, find the temperature distribution u (x, t) in the rod, where u (x, t) is governed by the partial differential equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$. **(8)**

Ans:

This is a binomial distribution with p = 0.6.

(i) Substituting p = 0.6, n = 5, x = 4 in the formula for the binomial distribution, we get $b(4,5,0.6) = \binom{5}{4}(0.6)^4(1-0.6)^{5-4} = 0.259.$

(ii) Similarly $b(5,5,0.6) = \binom{5}{5}(0.6)^5(1-0.6)^{5-5} = 0.078.$

So, the required probability is
b(4,5,0.6) + b(5,5,0.6) = 0.259 + 0.078 = 0.337.

Q.11 If the amount of a cosmic radiation to which a person is exposed while flying by jet across India is a random variable having the normal distribution with $\mu = 4.35$ mrem and $\sigma = 0.59$ mrem, find the probability that the amount of cosmic radiation to which a person will be exposed on such a flight is between 4.00 and 5.00 mrem. Given that F(1.10) = 0.8643, F(-0.59) = 0.2776, where F is the distribution function of the standard normal distribution. **(4)**

Ans:

Let X be the given random variable. Given that $\mu = 4.35$ and $\sigma = 0.59$.
Since X has the Normal distribution, the random variable $Z = \frac{X-\mu}{\sigma} = \frac{X-4.35}{0.59}$ has the standard normal distribution.
Let F be the density function of Z.
(i) The required probability is

$$F\left(\frac{5-4.35}{0.59}\right) - F\left(\frac{4-4.35}{0.59}\right)$$

$$= F(1.10) - F(-0.59) = 0.8643 - 0.2776 = 0.5867.$$

(ii) The required probability is

$$1 - F\left(\frac{5.5-4.35}{0.59}\right) = 1 - F(1.95) = 1 - 0.9774 = 0.0256.$$

Q.12 Let the probabilities that there are 0, 1, 2 and 3 power failures in a certain city during the month of July be respectively 0.4, 0.3, 0.2 and 0.1. Find the mean and variance of the number of power failures during the month of July in the city.

(4)

Ans:

The given data:

x	0	1	2	3
f(x)	0.4	0.3	0.2	0.1

Hence

$$\mu = \sum_{\text{all } x} x f(x) = 0.(0.4) + 1(0.3) + 2(0.2) + 3(0.1)$$

$$= 0.3 + 0.4 + 0.3 = 1$$

$$\sigma^2 = \sum_{\text{all } x} (x-\mu)^2 f(x) = (0-1)^2(0.4) + 0.(0.3) + 1.(0.2) + 4(0.1)$$

$$= 0.4 + 0 + 0.2 + 0.4 = 1.0.$$

Q.13 Show that $\vec{F}(x, y, z) = \left(2|x|y + z^3\right)\vec{i} + x|x|\vec{j} + 3xz^2\vec{k}$ is a conservative force field. Find its scalar potential and the work done in moving an object in this field from (1, -2, 1) to (3, 1, 4). **(8)**

Ans:

A necessary and sufficient condition that a force will be conservative is that $\nabla \times \vec{F} = 0.$

Now

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2|x|y + z^3 & x|x| & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\,[0-0] - \vec{j}\,[3z^2 - 3z^2] + \vec{k}\,[2|x| - 2|x|] = 0$$

Thus $\vec{F}$ is a conservative force field.

$$\vec{F} = \nabla\phi \text{ or } \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = (2|x|y + z^3)\,\vec{i} + x|x|\,\vec{j} + 3xz^2\,\vec{k}\,.$$

Therefore,

$$\frac{\partial\phi}{\partial x} = 2|x|y + z^3,\ \frac{\partial\phi}{\partial y} = x|x|,\ \frac{\partial\phi}{\partial z} = 3xz^2$$

Integrating,

$$\phi(x, y, z) = x|x|y + xz^3 + f(y, z)$$

$$\phi(x, y, z) = x|x|y + g(x, z)$$

$$\phi(x, y, z) = xz^3 + h(x, y)$$

These agree if we choose f (y, z) = 0, g (x, z) = x z^3 ,h (x, y) = x|x|y

so that $\phi(x, y, z) = x|x|y + xz^3 + c,$ for some constant c.

Then work done = $\phi(3, 1, 4) - \phi(1, -2, 1) = 202$

Q.14 Find the work done in moving the particle in the force field $\vec{F}(x, y, z) = 3x^2\vec{i} + (2xz - |y|)\vec{j} + z\vec{k}$ along the space curve $x^2 = 4y$, $3x^3 = 8z$ from x = 0 to x = 2. **(8)**

Ans: Parametric form of given space curve is

$x = 2t, \quad y = t^2, \quad z = 3t^3$ [2 marks]

Therefore $\vec{F} = 12t^2\,\vec{i} + (12t^4 - t^2)\,\vec{j} + 3t^3\,\vec{k}$

$$\vec{r}(t) = 2t\,\vec{i} + t^2\,\vec{j} + 3t^3\,\vec{k}$$

And $\vec{F}.d\vec{r} = (51t^5 - 2t^3 + 24t^2)dt$

$$\int_C \vec{F}.d\vec{r} = \int_0^1 (51t^5 - 2t^3 + 24t^2)dt = 16$$

Q.15 Evaluate $\iint_S \vec{F}.\vec{n}dA$, where $\vec{F}(x, y, z) = xy\vec{i} + z\vec{j} + 3yz\vec{k}$, S is the surface of the plane x + y + z = 1 in the first octant and $\vec{n}$ is unit outward normal to the surface S. **(8)**

Ans:

Let f (x, y, z) = x +y + z – 1 = 0 be the surface.

Then $\nabla f = \vec{i} + \vec{j} + \vec{k}$ and the unit outward normal is

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$$

Consider the projection of S on the xy plane. The projection of the portion of the plane in the first octant is the triangle bounded by x = 0, y =0 and x + y = 1.

We have

$$dA = \frac{dxdy}{\vec{n}.\vec{k}} = \sqrt{3}dxdy$$

$$\vec{F}.\vec{n} = \frac{1}{\sqrt{3}}(xy + z + 3yz)$$

Therefore,

$$\iint_S \vec{F}.\vec{n}\,dA = \iint_S (xy + z + 3yz)dxdy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x}(-2xy + 1 - x + 2y - 3y^2 dy)\right]dx \text{(by taking } z = 1 - x - y)$$

$$= \int_0^1 \left[-x(1-x)^2 + (1-x)^2 + (1-x)^2 - (1-x)^3\right]dx$$

$$= \int_0^1 (1-x)^2 dx = \frac{1}{3}.$$

Q.16 Verify the divergence theorem for $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ on the surface S of the sphere $x^2 + y^2 + z^2 = a^2$. **(8)**

Ans:

Gauss-divergence theorem states $\iint_S \vec{F}.\vec{n}\, dA = \iiint_V (\nabla.\vec{F})dV$

Where $\vec{n}$ is the unit outward normal to S.

The normal to the surface $x^2 + y^2 + z^2 - a^2 = 0$ is $\nabla S = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$ and the unit outward normal $\vec{n} = \frac{1}{a}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$

$$\text{and} \quad dA = \frac{dxdy}{\vec{n}.\vec{k}} = \frac{a\, dxdy}{z}$$

Therefore ,

$$\iint_s \vec{F}.\vec{n}\, dA = \iint_s \frac{\left(x^2 + y^2 + z^2\right)}{a} dA$$

$$= a^2 \iint_s \frac{dxdy}{z}$$

$$= a^2 \int_0^{2\pi}\int_0^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}} \text{ (taking } x = r\cos\theta, y = r\sin\theta)$$

$$= 2\pi a^3.$$

Now, $\nabla.\vec{F} = 3$. Therefore,

$$\iiint_v \nabla.\vec{F}dV = 3\iint_v dV = 2\pi a^3.$$

Q.17 Verify Stokes theorem for $\vec{F}(x, y, z) = z\vec{i} + x\vec{j} + z\vec{k}$, on the surface S of the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane. **(8)**

Ans:

The boundary C of S is the circle in the xy-plane of radius 3 and center at the origin.

Let $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\,\vec{k}, \quad 0 \le t \le 2\pi$

Be a parametric equation of C. Then

$$\oint \vec{F}.d\vec{r} = \int_0^{2\pi} 9\cos^2 t dt = 9\pi$$

Now, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & z \end{vmatrix} = \vec{j} + \vec{k}$

Now, $\vec{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{1}{3}\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)$

Hence $\left(\nabla\times\vec{F}\right).\vec{n} = \dfrac{1}{3}(y+z)$. Since the projection R of S on to the xy-plane is the circle $x^2 + y^2 = 9$. and $dA = \dfrac{dxdy}{\vec{n}.k} = \dfrac{3dxdy}{z}$,

we have $\iint_S \left(\nabla\times\vec{F}\right).\vec{n}\,dA = \iint_R \dfrac{y+z}{3}\dfrac{3dxdy}{z}$

$$= \iint_R \left(\frac{y}{z} + 1\right)dxdy$$

$$= \iint_R \frac{y}{\sqrt{9 - x^2 - y^2}}dxdy + \iint_R dxdy$$

$$= \int_{-3}^{3} y\left(\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{dx}{\sqrt{9 - x^2 - y^2}}\right)dy + 9\pi$$

$$= 0 + 9\pi = 9\pi.$$

Thus we get

$\oint_C \vec{F}.d\vec{r} = \iint_S (\nabla\times\vec{F}).\vec{n}\,dA,$

which verifies the Stokes theorem.

Q.18 Show that the function $Ln(z-i)$ is analytic everywhere except on the half line y =1 , x ≤ 0. **(8)**

Ans:

The function $Ln(z-i) = Ln(xiy - i)$ is not continuous,

Where

$I_m[x + i(y-1) = 0$ and

$R_e[x + i(y-1) \leq 0.$ we get $y = 1$ and $x \leq 0$.

Therefore, the function is single valued and continuous for all

$z = i + re^{i\theta}, r > 0, -\pi < \theta < \pi.$

The point z = I is the branch point and the line y = 1 is the branch cut.

We have

$$Ln(z-i) = \frac{1}{2}\ln(x^2 + (y-1)^2), i\tan^{-1}\left(\frac{y-1}{x}\right)$$

$$= u + iv.$$

Therefore, $u = \frac{1}{2}\ln(x^2 + (y-1)^2), \quad v = \tan^{-1}\left(\frac{y-1}{x}\right)$

we find that $\frac{\partial u}{\partial x} = \frac{x}{x^2 + (y-1)^2} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{y-1}{x^2 + (y-1)^2} = -\frac{\partial v}{\partial x}$.

The Cauchy-Riemann equations are satisfied. Since, the partial derivatives u_x, u_y, v_x, v_y are continuous. The given function is analytic everyone except on the half line y =1, $x \le 0$.

Q.19 If u is a harmonic function of two variables (x,y), then show that $w = u^2$ is not a harmonic function, unless u is a constant. **(8)**

Ans:

Since u is an harmonic function, $u_{xx} + u_{yy} = 0$.

Now, $w = u^2$ will be a harmonic function if

$w_{xx} + w_{yy} = 0$, we have $w_x = 2uu_x$, $w_y = 2uu_y$ and

$w_{xx} = 2(u_{x^2} + uu_{xx})$, $w_{yy} = 2(u_{y^2} + uu_{yy})$

Therefore,

$$\begin{aligned} w_{xx} + w_{yy} &= 2(u_{x^2} + uu_{xx} + u_{y^2} + uu_{yy}) \\ &= 2(u_{x^2} + u_{y^2}) + 2(uu_{xx} + uu_{yy}) \\ &= 2(u_{x^2} + u_{y^2}) \end{aligned}$$

Now, $w_{xx} + w_{yy} = 0$ when $u_x = 0$ and $u_y = 0$, that is u is a constant function.

Hence $w = u^2$ is not a harmonic function unless u is a constant.

Q.20 Evaluate the integral $I = \int_C \left(x + y^2 - ixy\right)dz$ from point 1-2i to point 2-i along the curve C, $C: z = z(t) = \begin{cases} t - 2i, & 1 \le t \le 2, \\ 2 - i(4 - t), & 2 \le t \le 3 \end{cases}$. **(8)**

Ans:

The curve C is continuous but not differentiable at z = 2, as

$$\frac{dz}{dt} = \begin{cases} 1 & 1 \le t \le 2 \\ i & 2 < t < 3. \end{cases}$$

Also $\frac{dz}{dt} \ne 0$ for any t . Therefore, the curve C is pieceuise smooth.

On the interval [1, 2], we have z = t – 2i, i – e x = t, y = -2.

$z'(t) = 1$, and $f(z) = x + y^2 - ixy = t + 4 + 2it$.

On the interval [2, 3], we have z = 2 – I (4 – t) i.e. x = 2, y = t - 4.

$z'(t) = i$ and $f(z) = 2 + (t-4)^2 + 2i(4-t)$

Hence,
$$\begin{aligned} I &= \int_c (x^2 + y - ixy)dz \\ &= \int_1^2 f(z).z'(t)dt + \int_2^3 f(z)\, z'(t)dt. \end{aligned}$$

$$I=\int_1^2 (4+t+2it)dt+\int_2^3 (2+(t-4)^2-2i(t-4)idt$$

$$=\left[4t+\frac{1}{2}(1+2i)t^2\right]_1^2+i\left[\frac{t^3}{3}-(4+i)t^2+(18+8i)t\right]_2^3$$

$$=\left(4+\frac{3}{2}-3\right)+i\left(3+\frac{19}{3}-2\right)$$

$$=\frac{5}{2}+\frac{22}{3}i.$$

Q.21 Find the residue of the function $f(z)=\cos(z+4)\cos\left(\frac{3}{z+2}\right)$ at $z=-2$. **(8)**

Ans:

The point z = -2 is an isolated essential singular point of f (z). The residue at z = -2 is the coefficient of $\frac{1}{z+2}$ in the Laurent series expansion of f (z) about z = -2. We write

$$f(z)=\cos\left(\frac{z^2+6z+5}{z+2}\right)=\cos\left(z+4-\frac{3}{z+2}\right)$$

$$=\sin(z+4)\sin\left(\frac{3}{z+2}\right)=\cos(z+2+2)\cos\left(\frac{3}{z+2}\right)+\sin(z+2+2)\sin\left(\frac{3}{z+2}\right)$$

$$=[\cos(z+2)\cos 2-\sin 2\sin(z+2)]\cos\left(\frac{3}{z+2}\right)+\sin\left(\frac{3}{z+2}\right)$$

$$[\sin(z+2)\cos 2+\cos(z+2)\sin 2]$$

$$=\cos 2\left[1-\frac{(z+2)^2}{2!}+\frac{(z+2)^4}{4!}---\right].\left[1-\frac{1}{2!}\left(\frac{3}{z+2}\right)^2+\frac{1}{4!}\left(\frac{3}{z+2}\right)^4---\right]$$

$$-\sin 2\left[(z+2)-\frac{(z+2)^3}{3!}---\right]\left[1-\frac{1}{2!}\left(\frac{3}{z+2}\right)^2+\frac{1}{4!}\left(\frac{3}{z+2}\right)^4---\right]$$

$$+\cos 2\left[(z+2)-\frac{(z+2)^3}{3!}---\right]\left[\left(\frac{3}{z+2}\right)-\frac{1}{3!}\left(\frac{3}{z+2}\right)^3---\right]$$

$$+\sin 2\left[1-\frac{(z+2)^2}{2!}+\frac{(z+2)^4}{4!}---\right]\left[\left(\frac{3}{z+2}\right)-\frac{1}{3!}\left(\frac{3}{z+2}\right)^3---\right]$$

We note that first and the third product do not contain $(z+2)^{-1}$ term. From the second and the fourth products, collecting the coefficients of $(z+2)^{-1}$, we obtain

$$\operatorname*{Res}_{z=-2} f(z)=-\sin 2\left[-\frac{3^2}{2!}-\frac{3^4}{3!4!}-\frac{3^6}{5!6!}---\right]+\sin 2\left[3-\frac{3^3}{2!3!}-\frac{3^5}{4!5!}---\right]$$

$$=\sin 2\left[\sum_{n=1}^{\infty}\frac{3^{2n}}{2n!(2n-1)!}+\sum_{n=0}^{\infty}\frac{3^{2n+1}}{2n\,!(2n+1)!}\right]$$

Q.22 Find all possible Taylor's and Laurent series expansions of the function $f(z)=\dfrac{1}{(z+1)(z+2)^2}$ about the point z = 1. **(10)**

Ans:

The given function is not analytic at the points z = -1 and z = -2. The distances between the point z = 1 and the points z = -1, z = -2 and 2 and 3 respectively. Therefore, we consider the regions (i) |z - 1| < 2 (ii) 2 < |z – 1| < 3 (iii) |z – 1| < 3. In the region, |z – 1| < 2, the function is analytic. Therefore, we obtain a Taylor's series expansion in this region. In the other regions, we obtain Laurent series expansions. We write

$$f(z)=\frac{1}{(z+1)(z+2)^2}=\frac{1}{(z+1)}-\frac{1}{(z+2)}-\frac{1}{(z+2)^2}$$

$$==\frac{1}{(z-1)+2}-\frac{1}{(z-1)+3}-\frac{1}{((z-1)+3)^2}$$

(i) In the region |z – 1| < 2, we write

$$f(z)=\frac{1}{2}\left[1+\frac{z-1}{2}\right]^{-1}-\frac{1}{3}\left[1+\frac{z-1}{3}\right]^{-1}-\frac{1}{9}\left[1+\frac{z-1}{3}\right]^{-2}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{3}\right)^n-\frac{1}{9}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-1}{3}\right)^n$$

$$=\sum_{n=0}^{\infty}(-1)^n\left[2^{-n-1}-3^{-n-1}-3^{-n-2}(n+1)\right](z-1)^n.$$

$$=\sum_{n=0}^{\infty}(-1)^n\left[2^{-n-1}-(n+4)3^{-n-2}\right](z-1)^n$$

The first series is valid in |z – 1| < 2 and the second and third series are rated in |z – 1| < 3. Hence the sum is valid in |z – 1| < 2.

(ii)In the region 2 < |z – 1| < 3, we have

$$f(z)=\frac{1}{z-1}\left[1+\frac{2}{z-1}\right]^{-1}-\frac{1}{3}\left[1+\frac{z-1}{3}\right]^{-1}-\frac{1}{9}\left[1+\frac{z-1}{3}\right]^{-2}$$

$$=\frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z-1}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{3}\right)^n-\frac{1}{9}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-1}{3}\right)^n$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^{n+1}}-\frac{1}{9}\sum_{n=0}^{\infty}(-1)^n(n+4)\left(\frac{z-1}{3}\right)^n.$$

The first series is valid in |z – 1| > 2, and Second series is valid in |z – 1| < 3. Hence the sum is valid in 2 < |z -1 | < 3.

(iii) In the region |z – 1| > 3, we have

$$f(z)=\frac{1}{z-1}\left[1+\frac{2}{z-1}\right]^{-1}-\frac{1}{z-1}\left[1+\frac{z-1}{3}\right]^{-1}-\frac{1}{(z-1)^2}\left[1+\frac{3}{z-1}\right]^{-2}$$

$$= \frac{1}{z\text{-}1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z-1}\right)^n - \frac{1}{z\text{-}1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z-1}\right)^n - \frac{1}{(z\text{-}1)^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{3}{z-1}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\frac{2^n - 3^n}{(z-1)^{n+1}} - \frac{3^n(n+1)}{(z-1)^{n+2}}\right].$$

Q.23 Evaluate the integral $\int_{-\infty}^{\infty}\frac{\sin^2 2x}{1+x^2}dx$ **(6)**

Ans:

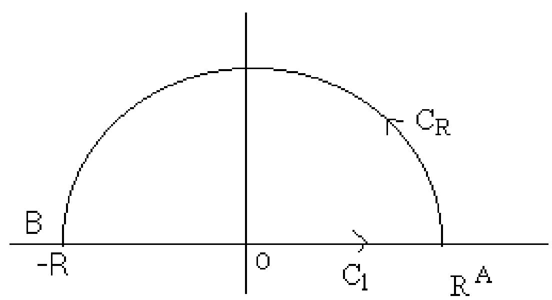
We have $\sin^2 2x = \frac{1}{2}(1-\cos 4x) = -R_e\left[\frac{1}{2}(e^{4ix}-1)\right]$

Therefore, $\int_{-\infty}^{\infty}\frac{\sin^2 2x}{1+x^2}dx = -\frac{1}{2}R_e\int_{-\infty}^{\infty}\frac{e^{4ix}-1}{1+x^2}dx.$

Consider the corresponding centre integral

$$I = \int_C \frac{e^{4iz}-1}{1+z^2}dz = \int_C (e^{4iz}-1)f(z)dz$$

where f (z) = $\frac{1}{1+z^2}$ and C is the path $C_R \bigcup C$, i.e. semicircle C_R from A to B and then C_1 from B to A along real axis.



The function f (z) = $\frac{1}{1+z^2}$ is analytic in the upper half plane except for the ample pole at z = i.

we find that

$$\operatorname*{Res}_{z=i}(e^{4iz}-1)f(z) = \lim_{z\to i}\left[\frac{(z-1)(e^{4iz}-1)}{1+z^2}\right] = \lim_{z\to i}\left(\frac{e^{4iz}-1}{z+i}\right) = \frac{e^{-4}-1}{2i}$$

we now write

$$I = \int_{C_R}\frac{e^{4iz}-1}{1+z^2}dz + \int_{-R}^{R}\frac{e^{4iz}-1}{1+z^2}dx = 2\pi i\left[\frac{e^{-4}-1}{2i}\right] = \pi(e^{-4}-1)$$

since $\left|e^{4iz}-1\right| \le \left|e^{4iz}\right| + 1 = e^{-4y} + 1 \le 2.$

Hence as $R \to \infty$.

$$\int_{-\infty}^{\infty} \frac{e^{4ix}-1}{x^2+1}dx = \pi(e^{-4}-1)$$

Therefore $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 2x}{1+x^2}dx = \frac{\pi}{2}(1-e^{-4}).$

Q.24 Find the directional derivative of the scalar point function $\varphi = x^2y + y^2z + z^2x$ at the point (2, 2, 2) in the direction of the normal to the surface $4x^2y + 2z^2 = 2$ at the point $(2,-1,3)$. **(6)**

Ans:

We have

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$= (2xy + z^2)\hat{i} + (2yz + x^2)\hat{j} + (2zx + y^2)\hat{k}$$

Hence $(\nabla\varphi)_{(2,2,2)} = 12(i + j + k)$

The given surface can be written as u = 2

Where u = $4x^2y + 2z^2$.

$\therefore \quad \nabla_u = 8xy\,\hat{i} + 4x^2\,\hat{j} + 4z\,\hat{k}.$

$(\nabla_u)_{(2,-1,3)} = -16\,\hat{i} + 16\,\hat{j} + 12\,\hat{k}.$

which is a vector along the normal to the surface at (2, -1, 3). Therefore repunned directional derivative is the component of $12(\hat{i} + \hat{j} + \hat{k})$ along $(-16\hat{i} + 16j + 12k)$.

$$= 12\hat{i} + 12j + 12k)\left(\frac{(-16\hat{i} + 16j + 12k)}{\sqrt{(-16)^2 + (16)^2 + (12)^2}}\right)$$

$$= \frac{36}{\sqrt{41}}.$$

Q.25 If $\bar{a}$ and $\bar{b}$ are constant vectors and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that $\nabla f(r) = \frac{f'(r)}{r}\bar{r}$ and hence show that $r^5\left[\bar{b}.\nabla\left(\bar{a}.\nabla\left(\frac{1}{r}\right)\right)\right] = 3(\bar{a}.\bar{r})(\bar{b}.\bar{r}) - (\bar{a}.\bar{b})r^2$, where $r = |\bar{r}|$ **(10)**

Ans:

We know that $\nabla f(r) = \frac{f'(r)}{r}.\bar{r}$ ---(1)

$$\Rightarrow \nabla\left(\frac{1}{r}\right) = \frac{1}{r}\left(-\frac{1}{r^2}\right)r = \frac{=1}{r^3}.r$$

$$a.\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}(a.r)$$

$$\therefore\ \nabla\left(a.\nabla\left(\frac{1}{r}\right)\right) = -\nabla\left(\frac{1}{r^3}(a.r)\right)$$

$$= -\nabla\left(\frac{1}{r^3}\right)(\bar{a}.\bar{r}) - \frac{1}{r^3}\nabla(\bar{a}.\bar{r})$$

Using (1) we get $\nabla\left(\frac{1}{r^3}\right) = \frac{1}{r}\frac{d}{dr}\left(\frac{1}{r^3}\right)\bar{r} = -\frac{3}{r^5}\bar{r}.$

Also we know that $\nabla(\bar{a}.\bar{r}) = \bar{a}$ if *a* is a constant vector.

$$\therefore \nabla\left(\bar{a}.\nabla\left(\frac{1}{r}\right)\right) = \frac{3}{r^5}\bar{r}(\bar{a}.\bar{r}) - \frac{1}{r^3}\bar{a}.$$

Therefore,

$$r^5\left[\bar{b}.\nabla\left\{\bar{a}.\nabla\left(\frac{1}{r}\right)\right\}\right]$$

$$= r^5\left[\frac{3}{r^5}(\bar{b}.\bar{r})(\bar{a}.\bar{r}) - \frac{1}{r^3}(\bar{b}.\bar{a})\right]$$

$$= 3(\bar{a}.\bar{r})(\bar{b}.\bar{r}) - (\bar{a}.\bar{b})r^2.$$

Q.26 Find the value of the surface integral $\iint_S \left(2x^2y\,dydz - y^2dzdx + 4xz^2dxdy\right)$ where S is the curved surface of the cylinder $y^2 + z^2 = 9$ bounded by the planes x = 0, x = 2. **(8)**

Ans:

We know that $\hat{n}\ d\bar{s}$ = idydz + jdzdx + kdxdy in terms of the projection of $d\bar{s}$ on the coordinate planes. Taking $\bar{F} = 2x^2y\hat{i} + 4xz^2\hat{k} - y^2\hat{j}$, the given integral can be written as $\iint_S \bar{F}.\hat{n}ds$.

To find $\hat{n}$, Let $\varphi = y^2 + z^2 - o$, then $\nabla\varphi = 2y\hat{j} + 2z\hat{k}$. .

Hence $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{y\hat{i} + z\hat{k}}{3}$.

$$\therefore\ \bar{F}.\hat{n} = -\frac{y^3}{3} + \frac{4}{3}xz^3.$$

Thus the given integral = $\frac{1}{3}\iint_S(-y^3 + 4xz^3)ds$.

Let $y = r\sin\theta,\ z = r\cos\theta,\ x = x$ so $ds = rd\theta dx$.

$$\therefore \quad \text{Integral} = \frac{1}{3}\iint_s (-r^3 \sin^3\theta + 4xr^3 \cos^3\theta) r dx d\theta. \quad (r = 3)$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^2 (-81 \sin^3\theta + 324x \cos^3\theta) dx d\theta.$$

$$= 27\int_0^{2\pi} (-x \sin^3\theta + 2x^2 \cos^3\theta)_0^2 d\theta.$$

$$= -54\int_0^{2\pi} \sin^3\theta \, d\theta + 216\int_0^{2\pi} \cos^3\theta \, d\theta = 0.$$

Q.27 The vector field $\bar{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ is defined over the volume of the cuboid given by $0 \le x \le a, 0 \le y \le b$, $0 \le z \le c$. Evaluate the surface integral $\iint_S \bar{F}\cdot d\bar{s}$, where S is the surface of the cuboid. **(8)**

Ans:

The surface integral $\iint_s \bar{F}.d\bar{s}$ has to be evaluated as the sum of six integrals corresponding to the six faces of the cuboid. Since S is a closed surface, the Gaurs divulgence thrm. is applicable.

Hence $\iint_s \bar{F}.d\bar{s} \iiint_v \nabla.\bar{F}\, dv.$

$$\text{div } \bar{F} = 2x + y.$$

$$\therefore \iiint_v \nabla.\bar{F}.dv = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (2x + y) dy\, dx.$$

$$= c\int_{x=0}^{a}\int_{y=0}^{b} (2x + y) dy\, dx.$$

$$= abc\left(a + \frac{b}{2}\right).$$

Q.28 A tightly stretched string with end points fixed at x = 0 and x = L, is initially at rest in equilibrium state. If it is set vibrating by giving to each of its points a velocity $\mu x(L - x)$, find the displacement of the string at any point x from one end, at any point of time t. **(12)**

Ans:

The partial differential equation for vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \qquad \ldots\ldots\ldots(1)$$

As per the boundary conditions provided, the form of solutions of (1) is

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4 \sin apt) \qquad \ldots\ldots\ldots(2)$$

Further, y (0, t) = 0 gives $o = (C_1 + 0)\ (c_3 \cos apt + c_4 \sin apt)$

$\Rightarrow C_1 = 0.$

$\therefore (2)$ becomes $y = C_2 \sin px(c_3 \cos apt + c_4 \sin apt).$

or $y = \sin px(c_3{}^1 \cos apt + c_4{}^1 \sin apt).$ …………(3)

Further, since the string is initially at rest,

$y(x,0) = 0 \quad \therefore (3)$ gives $C_3{}^1 = 0$

$\therefore \quad y = C_4{}^1 \sin px \sin apt.$

Also from the condition y (*l*, t) = 0, we get

$o = C_4{}^1 \sin pl \sin apt.$ ………..(4)

which gives $\sin pl = 0 \rightarrow pl = n\pi.$

$$\therefore \quad p = \frac{n\pi}{L}. \quad n = 1, 2, ---$$

$$\therefore (4) \text{ becomes} \quad y = C_4{}^1 \sin \frac{n\pi x}{L} \sin \frac{an\pi}{L} t, \quad n = 1, 2, ---$$

Thus the most general solution can be written as

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x . \sin\left(\frac{an\pi}{L} t\right). \qquad \text{…………(5)}$$

$$\therefore \quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x . \cos\left(\frac{an\pi}{L} t\right)\left(\frac{an}{L}\pi\right)$$

From the boundary condition $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \mu x(L - x),$ we get

$$\mu x(L - r) = \sum_{n=1}^{\infty} b_n \left(\frac{an\pi}{L}\right) \sin \frac{n\pi}{L} x. \qquad \text{…………(6)}$$

To determine b_n we expand $\mu x(L - x)$ in a half range Fourier sine series in (0, L), we get

$$\mu x(L - x) = \sum_{n=1}^{\infty} b_n{}^1 \sin \frac{n\pi}{L} x \qquad \text{…………(7)}$$

where $b_n{}^1 = \frac{2}{L}\int_0^l \mu x(L - x) \sin \frac{n\pi}{L} x\, dx = \frac{4\mu L^2}{n^3\pi^3}[1 - (-1)^n]$

comparing (6) and (7) yields

$$\frac{an\pi}{L} b_n = b_n{}^1 = \frac{4\mu L^2}{n^3\pi^3}[1 - (-1)^n]$$

$$\Rightarrow \quad b_n = \begin{cases} 0, & n = \text{even} \\ \dfrac{8\mu L^3}{an^4\pi^4}, & n = \text{odd.} \end{cases}$$

Thus

$$y = \frac{8\mu L^3}{an^4\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2n-1)a\pi t}{L}.$$

Q.29 Evaluate the integral $\oint_C \frac{dz}{2-\overline{z}}$; where $C:|z|=1$ **(4)**

Ans:

$$2-\overline{Z} = 2-\frac{Z\overline{Z}}{Z} = 2-\frac{|Z|^2}{Z} = 2-\frac{1}{Z} = \frac{2Z-1}{Z}.$$

$$\therefore I = \oint_C \frac{dZ}{2-\overline{Z}} = \oint \frac{Z}{2Z-1}dZ = \frac{1}{2}\oint_C \frac{Z}{(Z-\frac{1}{2})}dZ.$$

The integrand $\frac{Z}{Z-\frac{1}{2}}$ is not analytic at the point Z = ½ which lies within C. Using Cauchy integrand formula $\frac{1}{2}\oint_C \frac{ZdZ}{Z-\frac{1}{2}} = \frac{1}{2}2\pi i f\left(\frac{1}{2}\right) = \frac{\pi i}{2}.$

Q.30 A continuous type random variable X has probability density f(x) which is proportional to x^2 and X takes values in the interval [0, 2]. Find the distribution function of the random variable use this to find P (X >1.2) and conditional probability P(X > 1.2/ X>1). **(8)**

Ans:

Suppose there are 100 bank account holders. So, 20 persons have taken loans among 20, 18 are males and 2 females. Among 80, who are not loan takers, 76 are males and 4 females. So total males are 94 and females are 6 among account holders.

Males who have taken loans = 18.

Totals male accounts holders = 94.

So, the probability of an account holders who is randomly selected turns out a male that he has taken loans with the bank = $p = \frac{18}{94} = 0.1915$.

Q.31 Suppose that on an average 1 house in 1000 houses gets fire in a year in a district. If there are 2000 houses in that district find the probability that exactly 5 houses will have fire during the year. Also find approximate probability using Poisson distribution. **(8)**

Ans:

In Poison distribution,N = 2000, p = 1/1000

Mean = m = np =2

$$\therefore \quad f(x) = \frac{e^{-m}.\, m^x}{x!} = \frac{e^{-2}Z^x}{x!}.$$

$$\therefore \quad P(5) = \frac{e^{-2}.\, 2^5}{5!}.$$

Q.32 Derive the mean and variance of binomial distribution. **(8)**

Ans:

$$f(x)={}^{n}C_{x}p^{x}(1-p)^{n-x}$$

$$\text{Mean} = E(x)=\sum_{x=0}^{\infty} x.{}^{n}C_{x}p^{x}q^{n-x}$$

$$=\sum_{0}^{n}\frac{x.n!}{(n-x)!.x!}p^{x}q^{n-x}$$

$$= np\sum_{0}^{n}\frac{(n-1)!}{(x-1)!(n-x)!}p^{x-1}q^{n-x}$$

$$= np\sum_{n=1}^{\infty}{}^{n-1}C_{x-1}\,p^{x-1}q^{n-x}$$

$$= np(q+p)^{n-1}=np.$$

$$\text{Variance} = E(x^{2})-(E(x))^{2}$$

$$=E[x(x-1)] + E(x)-(E(x))^{2}$$

$$E(x(x-1))=\sum_{0}^{n} x(x-1)\,.\,{}^{n}C_{x}\,p^{x}q^{n-x}$$

$$=\sum_{2}^{n}\frac{n!}{(x-2)!(n-x)!}\,.\,p^{x}q^{n-x}$$

$$= n(n-1)p^{2}\sum_{2}^{n}{}^{n-2}C_{x-2}\,p^{x-2}q^{n-x}$$

$$= n(n-1)p^{2}.$$

$$\therefore \text{variance} = n(n-1)p^{2}+np-n^{2}p^{2}=np-np^{2}=npq.$$

Q.33 Determine the analytic function f (z) = u +i v, given that $3u+2v=y^{2}-x^{2}+16xy$.**(8)**

Ans:

It is given that $3u+2v=y^2-x^2+16xy$, thus differentiating partially w.r.t.x and y

$$3\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial x}=-2x+16y, \qquad 3u_x+2v_x=-2x+16y$$

$$3\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial y}=2y+16x \qquad \text{or} \qquad 2u_x-3v_x=2y+16x$$

Solving, we get $u_x=2x+4y$ and $v_x=-4x+2y$

Thus $f'(z)= u_x+ iv_x=2x+4y+i(2y-4x)$

By Milne's Thomson method, putting x=z and y=0, we get

$f'(z)=2(1-2i)z$ Thus $f(z) = (1-2i)z^2+c$.

Q.34 If w = u + i v is an analytic function, then show that the family of curves u (x, y) = a, cut the family of curves v (x, y) = b orthogonally, a, b being parameters. **(6)**

Ans:
Let $w=u+iv$, and

$$\phi_1 = u(x,y) - a = 0, \quad \phi_2 = v(x,y) - b = 0$$

$$\bar{\nabla}\phi_1 = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} \qquad \nabla\phi_2 = \frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j}$$

$$\bar{\nabla}\phi_1 \cdot \bar{\nabla}\phi_2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = u_x v_x + u_y v_y$$

Since w is an analytic function, thus $u_x = v_y$ and $u_y = -v_x$

Thus $\bar{\nabla}\phi_1 \cdot \bar{\nabla}\phi_2 = v_x v_y - v_x v_y = 0$. Hence ϕ_1, ϕ_2 cut orthogonally.

Q.35 Find the image of infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$, under the mapping $w = \frac{1}{z}$. **(7)**

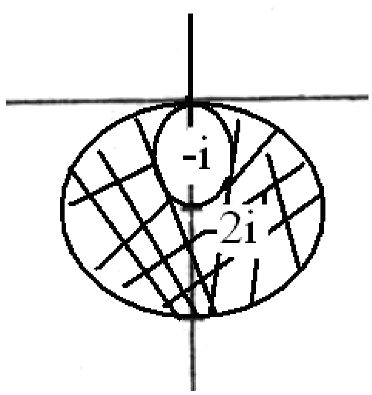
Ans:
Let w = 1/z, then z = 1/w. Thus

$$x + iy = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2} \Rightarrow x = \frac{u}{u^2+v^2}, \ y = \frac{-v}{u^2+v^2}$$

Since $y \le 1/2$, thus $u^2+ v^2+2v \ge 0$ or $u^2+ (v+1)^2 \ge 1$ i.e. $|w+i| \ge 1$

The boundary of this region is the outside of the circle with centre at(0,-1) and radius 1,The region $y \ge 1/4$ is transformed to

$$\frac{-v}{u^2+v^2} \ge \frac{1}{4} \Rightarrow u + (v+2)^2 \le 4 \Rightarrow |w+2i| \le 2,$$



The boundary of this region is the outside of the circle with centre at (0,-2i) and radius 2. Hence, the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ maps into inside of the circle $|w+2i| = 2$ and outside of the circle $|w+i| = 1$. See the shaded region in the figure.

Q.36 Find the linear fractional transformation that maps the points i, -1, 1 of z-plane into the points 0, 1, ∞ of w-plane respectively. Where in w-plane is the interior of unit disc $|z| \le 1$ mapped by the fractional transformation obtained? **(7)**

Ans:
Since, $w_1=0$, $w_2=1$, $w_3=\infty$, and $z_1 = i$, $z_2=-1$, $z_3=1$.

The bilinear transformation is given by

$$\frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{\left(\frac{z}{z_3}-1\right)}\frac{\left(\frac{z_2}{z_3}-1\right)}{(z_2-z_1)} \quad => \quad w=\left(\frac{2}{1+i}\right)\frac{z-i}{z-1} \quad => \quad w=(1-i)\frac{z-i}{z-1}.$$

Solving for z gives

$$z=\frac{w-1-i}{w-1+i}.$$

$$|z|<1, \;\Rightarrow \left[(u-1)^2+(v-1)^2\right] < (u-1)^2+(v+1)^2$$

$$\Rightarrow v=0 \;\; or \;\; \operatorname{Im} w > 0.$$

Thus interior of the circle $|z| \le 1$, in z plane is mapped onto the Imw > 0.

Q.37 Show that $\vec{F} = \left(y^2+2xz^2\right)\hat{i} + (2xy - z)\hat{j} + \left(2x^2z - y + 2z\right)\hat{k}$ is irrotational and hence find its scalar potential. **(8)**

Ans:
It is given that

$$Curl\ \overline{F} = \overline{\nabla}\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+2xz^2 & 2xy-z & 2x^2z-y+2z \end{vmatrix}$$

$= \hat{i}(1-1) - \hat{j}(4xz-4xz) + \hat{k}(2y-2y) = 0$ $\therefore$ Given vector is irrotational. Thus it can be expressed as $\overline{F} = \overline{\nabla}\phi$ where ϕ is scalar function.

$$\therefore \quad \frac{\partial \phi}{\partial x} = y^2+2xz^2, \qquad \frac{\partial \phi}{\partial y} = 2xy - z, \qquad \frac{\partial \phi}{\partial z} = 2x^2z - y + 2z$$

Integrating w.r.t. x, y, z we get

$$\phi = \begin{cases} y^2x + x^2z^2 + f_1(y,z) \\ xy^2 - zy + f_2(x,z) \\ x^2z^2 + z^2 - zy + f_3(x,y) \end{cases}$$

Since these three must be equal

$\therefore \phi = x^2z^2 + xy^2 - yz + z^2$ +c

Q.38 Find the directional derivative of the scalar function $\phi = xy^2 + yz^2$ at the point (2, -1, 1) in the direction of the normal to the surface $x(\ell nz) - y^2 + 4 = 0$ at the point (-1, 2, 1). **(6)**

Ans:

$\overline{\nabla}\phi = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k} \Rightarrow \overline{\nabla}\phi_{(2,-1,1)} = \hat{i} - 3\hat{j} - 2\hat{k}$

A vector normal to the surface xln(z) - y^2 + 4 = 0 is given by

$\ln z\hat{i} - 2y\hat{j} + \dfrac{x}{z}\hat{k}$, which at point (-1,2,1) becomes $-4\hat{j} - \hat{k}$. The required directional derivative is the component of $(\hat{i} - 3\hat{j} - 2\hat{k})$ along $-4\hat{j} - \hat{k}$,

$$= (\hat{i} - 3\hat{j} - 2\hat{k}) \cdot \left[\frac{-4\hat{j} - \hat{k}}{\sqrt{17}}\right] = \frac{12+2}{\sqrt{17}} = \frac{14}{\sqrt{17}}.$$

Q.39 Find the work done by a force $\overrightarrow{F} - \sin y\,\hat{i} + x(1 + \cos\ y)\hat{j} + z\,\hat{k}$ by moving a particle once around the circle $x^2 + y^2 = a^2$. **(7)**

Ans:

$\overline{F} = \sin y\hat{i} + x(1 + \cos y)\,\hat{j} + z\hat{k}$

At C: $x^2 + y^2 = a^2$, z=0, thus $\overline{F} = \sin y\hat{i} + x(1 + \cos y)\,\hat{j}$.

Work = $\int_C \overline{F} \cdot d\overline{R} = \int_C [\sin y dx + x(1 + \cos y) dy] = \iint_R (1 + \cos y - \cos y) dxdy$

where R is the region bounded by circle $x^2 + y^2 = a^2$. Let x=rcosθ, y=rsinθ, then dxdy=rdrdθ , where r changes from 0 to a and θ changes from 0 to 2π.

Thus work = $\int_0^a \int_0^{2\pi} r dr d\theta = \pi a^2$.

Q.40 Show that the vector field $\overrightarrow{F} = (yze^{xyz} - 4x)\hat{i} + (xze^{xyz} + z)\hat{j} + (xye^{xyz} + y)\hat{k}$ is conservative. Hence evaluate the line integral

$\int \left(yze^{xyz} - 4x\right)dx + \left(xze^{xyz} + z\right)dy + \left(xye^{xyz} + y\right)dz$ along a path joining the points (0, 0, 0) to (1, 1, 1) **(7)**

Ans:

$$Curl\ \overline{F} = \overline{\nabla} \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xyz} - 4x & xze^{xyz} + z & yxe^{xyz} + y \end{vmatrix}$$

$$= \hat{i}\left(xe^{xyz} + 1 + xyxze^{xyz} - xe^{xyz} - 1 - xzxye^{xyz}\right)$$
$$- \hat{j}\left(ye^{xyz} + xyyze^{xyz} - ye^{xyz} - xyyze^{xyz}\right)$$
$$+ \hat{k}\left(ze^{xyz} + xzyze^{xyz} - ze^{xyz} - yzxze^{xyz}\right) = 0$$

$\therefore$ Given vector represents a conservative field. Thus it can be expressed as $\overline{F} = \overline{\nabla}\phi$ where ϕ is scalar function.

$$\therefore \quad \frac{\partial \phi}{\partial x} = yze^{xyz} - 4x, \qquad \frac{\partial \phi}{\partial y} = xze^{xyz} + z, \qquad \frac{\partial \phi}{\partial z} = xye^{xyz} + y$$

Integrating w.r.t. x, y, z we get

$$\phi = \begin{cases} e^{xyz} - 2x^2 + f_1(y, z) \\ e^{xyz} + zy + f_2(x, z) \\ e^{xyz} + zy + f_3(x, y) \end{cases}$$

Since these three must be equal

$\therefore \phi = e^{xyz} - 2x^2 + yz + c$ Also, $\int \overline{F} \cdot d\overline{r} = \int_{(0,0,0)}^{(1,1,1)} \overline{\nabla}\phi \cdot dr = [\phi]_{(0,0,0)}^{(1,1,1)} = e - 2.$

Q.41 A rod of length ℓ has its lateral surface insulated and is so thin that heat flow in the rod can be regarded as one dimensional. Initially the rod is at the temperature 100 throughout. At t=0 the temperature at the left end of the rod is suddenly reduced to 50 and maintained thereafter at this value, while the right end is maintained at 100. Let u (x, t) be the temperature at point x in the rod at any subsequent time t.

a. Write down the appropriate partial differential equation for u (x, t), with initial and boundary conditions.
b. Solve the differential equation in (i) above using method of separation of variables and show that $u(x, t) = 50 + \frac{50x}{\ell} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{\ell}\exp\frac{-n^2\pi^2 t}{a^2\ell^2}$

Where a^2 is the constant involved in the partial differential equation. **(3+11)**

Ans:

(i) Let the equation for conduction of heat be

$$a^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad - \qquad (1)$$

Prior to temperature change at end B, when t = 0, the heat flow was independent of time (steady state condition), when u depends only on x i.e.

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad => \qquad u = ax + b$$

Since u = 100 for x = 0 and x = L

$\therefore$ b = 100 and a = 0.

Thus initial condition is expressed as

$$u(x,0) = 100 \quad - \quad (2)$$

The boundary conditions are

$$\left.\begin{aligned} u(0,t) &= 50 \\ u(L,t) &= 100 \end{aligned}\right\} \forall t \qquad - \qquad (3)$$

(ii) Assuming product solution u(x,t) = X(x). T(t) and substituting in equation (1), we get $\frac{X''}{X} = a^2 \frac{T'}{T} = \mu \quad (say) \quad (4)$

Case I : If $0 < \mu = \lambda^2,\ \lambda > 0,$ (4) gives $X'' - \lambda^2 X = 0 \ \ and \ \ T' = \frac{\lambda^2}{a^2} T.$

Solving we have the solution

$$u(x,t) = (A\cosh \lambda x + B\sinh \lambda x)Ce^{\frac{\lambda^2}{a^2}t}.$$

This solution is rejected as exponential term makes temperature u(x,t) increases without bounds as t $\rightarrow \infty$.

Case II : If μ = 0. (4) gives

$X'' = 0 \ \ and \ \ T' = 0$

Integrating, we obtain X = (A X + B) and T = C.

Thus we can write u(x,t) = (A_1 X + B_1), where A_1 = AC and B_1 =BC are arbitrary constant. Using boundary conditions (3), we get $u_0(x,t) = 50 + \frac{50}{l}x \quad - \quad (5)$ is a solution of heat equation.

Case III : If $\mu < 0, say\ -\lambda^2,\ \lambda > 0,$ then from (4) (as in case (I)) we conclude that

$$u(x,t) = (A\cos \lambda x + B\sin \lambda x)Ce^{-\frac{\lambda^2}{a^2}t} \qquad (6)$$

Since we already have a solution (5) satisfying boundary conditions (3) we can find A, B in (6) by satisfying the condition u(0,t) = 0 =u(l,t) which gives A = 0,

B sinλl = 0 .

As B = 0 leads to trivial solution we must have sinλl = 0 or $\lambda = \frac{n\pi}{l}$, n =1,2,…..

Combining (5) and (6), we have

$$u(x,t) = 50 + \frac{50}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l} x e^{-\frac{n^2\pi^2 t}{a^2 l^2}} \qquad (7)$$

as a general solution of (1).
Applying initial condition (2) to the general solution we must have

$$u(x,0) = 100 = 50 + \frac{50}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l} x$$

implies that b_n are the coefficient in half range sine series expansion of

$$100 - \left(50 + \frac{50}{l}x\right) = 50 - \frac{50}{l}x$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l \left(50 - \frac{50}{l}x\right)\sin\frac{n\pi x}{l}dx \quad \Rightarrow b_n = \frac{100}{n\pi}$$

Putting b_n in (7) we get required solution.

Q.42 Evaluate the complex integral $\varphi(a) = \int_{|z-3|=1} \frac{1-\cos 2(z-a)}{(z-a)^3} dz, a \neq 3$.

Also find $\varphi'\left(3 + \frac{i}{2}\right)$. **(6)**

Ans:

$$\phi(a) = \int_C \frac{1-\cos 2(z-a)}{(z-a)^3} dz \qquad a \neq 3$$

If $|a-3| > 1$, then f(z) $= \frac{1-\cos 2(z-a)}{(z-a)^3}$ is analytic within and on C. Thus $\phi(a)$ =0, by Cauchy Theorem if $|a-3| > 1$, .

If $|a-3| < 1$, the singularity of f(z) $= \frac{1-\cos 2(z-a)}{(z-a)^3}$ lies within C and by Cauchy integral formula $\phi(a) = 2\pi i \operatorname{Res}_{z=a} f(z)$

Since $f(z) = \frac{2}{z-a} - \frac{16}{4!}(z-a) + ...$, $\operatorname{Res}_{z=a} f(z) = 2$

$\therefore \phi(a) = 4\pi i,\ if\ |a-3| < 1$. Since $\phi(a)$ is constant, then $\phi'(3 + \frac{i}{2}) = 0$.

Q.43 Find the residues of $f(z) = \frac{z^2}{(z-1)(z+2)^2}$ at its isolated singularities, using Laurent's series expansions. **(8)**

Ans:

f(z) $= \frac{z^2}{(z-1)(z+2)^2}$, z=1, is apole of order 1 and z=-2 is a pole of order 2.

<u>Expanding about z=1,</u>

let z-1=t, i.e. z=t+1,

$$\therefore f(t) = \frac{(t+1)^2}{t(t+3)^2} = \frac{1}{9t}(t^2+2t+1)(1+\frac{t}{3})^{-2},\ 0<|t|<3$$

$$\therefore f(t) = \frac{1}{9}\left[t+2+\frac{1}{t}\right]\left[1-\frac{2t}{3}+\frac{3t^2}{9}-.....\right] = \frac{1}{9}\left[\frac{4}{3}+\frac{1}{t}-\frac{t}{3}-\frac{2t^2}{3}.....\right]$$

Since there is only one term in negative powers of (z-1), therefore z = 1, is a pole of order 1. Residue at z = 1 is the coefficient of 1/t, which is 1/9.

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{9}.$$

Expanding about z=-2,

let z+2=t, i.e. z=t-2,

$$\therefore f(t) = \frac{(t-2)^2}{t^2}(t-3)^{-1} = -\frac{1}{3}(1-\frac{4}{t}+\frac{4}{t^2})(1-\frac{t}{3})^{-1}$$

$$\therefore f(t) = -\frac{1}{3}\left[\frac{1}{9}-\frac{8}{3t}+\frac{4}{t^2}-\frac{t}{9}.....\right],\quad 0<|t|<3$$

Since there are only two terms in negative powers of (z+2), therefore z = -2, is a pole of order 2. Residue is the coefficient of 1/t, which is 8/9.

$$\operatorname*{Res}_{z=-2} f(z) = \frac{8}{9}.$$

Q.44 Let u (x, y) be continuous with continuous first and second partial derivatives on a simple closed path C and throughout the interior D of C. Show that $\iint_D \nabla^2 u \; dA = \int_c \frac{du}{dn} ds$ where $\frac{du}{dn}$ is the directional derivative of u along the outer normal to the curve C. **(6)**

Ans: Let the position vector of a point on C, in terms of arc length s be $\bar{r}(s) = x(s)\hat{i} + y(s)\hat{j}$. Then the tangent vector to C is given by

$\bar{T} = \frac{d\bar{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$ and a normal vector $\hat{n}$ is given by

$\hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$. Thus

$$\int_C \frac{\partial u}{\partial n} ds = \int_C \bar{\nabla} u.\hat{n} ds,$$

since $\frac{\partial u}{\partial n}$ is a directional derivative of u in the direction of $\hat{n}$. Now, using Green's theorem, we obtain

$$\int_C \frac{\partial u}{\partial n} ds = \int_C \left(\frac{\partial u}{\partial x}\frac{dy}{ds} - \frac{\partial u}{\partial y}\frac{dx}{ds}\right) ds = \int_C \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right)$$

$$= \iint_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy = \iint_R \nabla^2 u dxdy$$

Q.45 Verify Gauss divergence theorem for $\vec{F} = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$ on the surface

S of the cuboid formed by the planes x = 0, x = a, y = 0, y = b, z = 0 and z = c. **(8)**

Ans:

$$div\,\overline{F} = 2x + 2y + 2z$$

$$\therefore \int_R div\,\overline{F}dv = 2\int_0^a\int_0^b\int_0^c (x+y+z)dxdydz$$

$$= 2\int_0^c\int_0^b\left[\frac{x^2}{2} + xy + zx\right]_0^a dydz = 2\int_0^c dz\int_0^b\left[\frac{a^2}{2} + ay + za\right]_0^a dydz$$

$$= 2\int_0^c\left[\frac{ya^2}{2} + \frac{ay^2}{2} + ayz\right]_0^b dz = 2\int_0^c\left[\frac{ba^2}{2} + \frac{ab^2}{2} + abz\right]dz = abc(a+b+c)$$

Also, $\int_S \overline{F}.\overline{n}\,ds = \int_{S_1+S_2+S_3+S_4+S_5+S_6} \overline{F}.\overline{n}\,ds$,

where S_1, S_2, S_3, S_4, S_5 and S_6 are the six faces of the cuboid.

On S_1, $z = 0,\ \overline{n} = -\hat{k}$ $\therefore \int_{S_1}\overline{F}.\overline{n}\,ds = 0$

On S_2 , $z = c,\ \overline{n} = \hat{k}$, *i.e.* $\therefore \int_{S_2}\overline{F}.\overline{n}\,ds = \int_0^a\int_0^b c^2dxdy = abc^2$

On S_3, $y = 0,\ \overline{n} = -\hat{j}$ $\therefore \int_{S_3}\overline{F}.\overline{n}\,ds = 0$

On S_4, $y = b,\ \overline{n} = \hat{j}$ $\therefore \int_{S_4}\overline{F}.\overline{n}\,ds = \int_0^a\int_0^c b^2dxdz = acb^2$

On S_5, $x = 0,\ \overline{n} = -\hat{i}$, $\therefore \int_{S_5}\overline{F}.\overline{n}\,ds = 0$

On S_6, $x = a,\ \overline{n} = \hat{i}$ $\therefore \int_{S_6}\overline{F}.\overline{n}\,ds = \int_0^c\int_0^b a^2dzdy = a^2$

Thus $\therefore \int_S \overline{F}.\overline{n}\,ds = abc(a+b+c) = \int_R div\overline{F}\,dv$

Hence Gauss Divergence theorem is verified.

Q.46 The probability of an airplane engine failure (independent of other engines) when the aircraft is in flight is (1-P). For a successful flight at least 50% of the airplane engines should remain operational. For which values of P would you prefer a four engine airplane to a two engine one? **(7)**

Ans:

Let X be the number of engines that do not fail and let S_k denote the successful flight with k engine plane. Let 1-p=q,

$P(S_2) = P(X \geq 1) = 1-P(X=0) = 1-q^2, \because P(X=r) = \dfrac{n!}{r!(n-r)!} p^r q^{n-r}$

$P(S_4) = P(X \geq 2) = 1-P(X=0)-P(X=1) = 1-4q^3+3q^4.$

For $P(S_4) > P(S_2)$, we have

$1-4q^3+3q^4 > 1-q^2$ or $q^2(1-q)(1-3q) > 0.$

If $0<q<1/3$, i.e. $2/3<p<1$, the four engine plane is preferred.

Q.47 If the resistance X of certain wires in an electrical networks have a normal distribution with mean of 0.01 ohm and a standard deviation of 0.001 ohm, and specification requires that the wires should have resistance between 0.009 ohm and 0.011 ohms, then find the expected number of wires in a sample of 1000 that are within the specification. Also find the expected number among 1000 wires that cross the upper specification.
(You may use normal table values $\Phi(.5) = .6915, \Phi(1) = .8413, \Phi(1.5) = .9332, \Phi(2) = .9772$). **(7)**

Ans:

Given that

$\mu = 0.01, \quad \sigma = 0.001$

(i) $P(0.009 \leq X \leq 0.011) = P\left(\dfrac{0.009-0.01}{0.001} \leq \dfrac{X-0.01}{0.001} \leq \dfrac{0.011-0.01}{0.001}\right) = P(-1 \leq z \leq 1)$

$= \phi(1) - \phi(-1) = 2\phi(1) - 1 = 2(0.8413) - 1 = 0.6826$

Expected number of wires in a sample of 1000 with this specification
= 1000(0.6826)=682.6=683 approximately.

(ii) $P(X \geq 0.011) = P(z \geq 1) = 1 - P(z < 1) = -\phi(1) + 1 = 0.1587$

Hence, expected number among 1000 wires that cross the upper specification = 1000(0.1587)=158.7=159 approximately.

Q.48 Suppose that certain bolts have length $L = 400 + X$ mm, where X is a random variable with probability distribution function.

$f(x) = \dfrac{3}{4}(1-x^2), \quad -1 \leq x \leq 1$ and 0, otherwise

(i) Determine C so that with probability $\dfrac{11}{16}$, a bolt will have length between 400 – C and 400 + C.

(ii) Find the mean and variance of bolt length L. Also find mean and variance of (2 L+5). **(4+10)**

Ans:

(i) $P(400-C \leq L \leq 400+C) = 11/16$

$\Rightarrow$ P(-C $\le$ L-400 $\le$ C)= $\frac{11}{16}$ $\Rightarrow$ P(-C $\le$ X $\le$ C)= $\frac{11}{16}$ $\Rightarrow F$(C)-F(-C)= $\frac{11}{16}$

$$\Rightarrow \int_{-1}^{C} \frac{3}{4}(1-x^2)dx - \int_{-1}^{-C} \frac{3}{4}(1-x^2)dx = \frac{11}{16} \quad \Rightarrow \int_{-C}^{C} \frac{3}{4}(1-x^2)dx = \frac{11}{16}$$

$$\Rightarrow \frac{3}{4}\left[2C - \frac{2C^3}{3}\right] = \frac{11}{16} \quad \Rightarrow 8C^3 - 24C + 11 = 0$$

$$\Rightarrow C = \frac{1}{2}, \frac{-1 \pm 3\sqrt{5}}{4}$$

Since C$\neq \frac{-1 \pm 3\sqrt{5}}{4}$ as it is either >1 or <1. Thus C=1/2.

(ii) E(L) = E(400+X) = 400+E(X)

$$= 400 + \int_{-1}^{1} \frac{3}{4} x(1-x^2)dx = 400 \quad \left\{ \because \int_{-1}^{1} \frac{3}{4} x(1-x^2)dx = 0 \, being\ an\ odd\ function \right\}$$

Thus E(L) = 400.
V(L) = V(400+X) = V(X) = E(X^2), as E(X) = 0

$$= \int_{-1}^{1} \frac{3}{4} x^2(1-x^2)dx = \int_{0}^{1} \frac{3}{2} x^2(1-x^2)dx = \frac{1}{5}$$

Thus E(L) = 400, V(L) = 0.2. Therefore,
E(2L+5) = 2E(L) + 5=805
V(2L+5) = 4V(L) = 0.8.

Q.49 Evaluate the integral $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2}$ dx, using contour integration. **(7)**

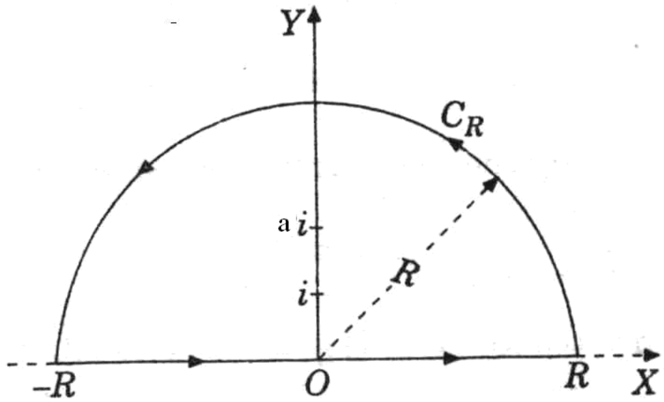
Ans:

$$Let\ I = \int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$$

Since integrand is an even function, thus

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\text{Im}\, x e^{ix}}{x^2 + a^2} dx$$

Consider the contour integral $I = \int_C f(z) e^{iz} dz, \quad where\ f(z) = \frac{z}{z^2 + a^2}$ and C is the path from –R to R along the real axis and from R to –R along C_R . Now f(z) is analytic in upper half of the plane except at z=a*i*, which is pole of order 1.



Residue of f(z) at $z = ai$ $= \lim_{z\to ai} \dfrac{(z-ai)ze^{iz}}{(z-ai)(z+ai)} = \dfrac{e^{-a}}{2}$

$\therefore I = \int_{C_R} f(z)e^{iz}dz + \int_{-R}^{R} f(x)e^{ix}dx = \pi e^{-a} i$ by Residue theorem.

Now, $\left|\dfrac{z}{z^2+a^2}\right| \le \dfrac{R}{R^2-a^2} \to 0 \; as \; R\to\infty$. Therefore, by Jordan's Lemma

$$\int_{C_R} \frac{ze^{iz}}{z^2+a^2}dz \to 0 \; as \; R \to \infty$$

$$\therefore \int_{-\infty}^{\infty} f(x)e^{ix}dx = \pi e^{-a} i$$

Equating imaginary part, we get

$$I = \int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx = \frac{\pi e^{-a}}{2}$$

Q.50 Prove that $\text{grad}\left(\text{div}\ \vec{F}\right) = \text{curl}\left(\text{curl}\ \vec{F}\right) + \nabla^2 \vec{F}$. **(7)**

Ans:

$$curl(curl\overline{F}) = \overline{\nabla}\times\left(\overline{\nabla}\times\overline{F}\right) = \left(\sum \hat{i}\frac{\partial}{\partial x}\right)\times\left(\sum \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\right)$$

$$= \sum \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right] = \sum \hat{i}\left[\frac{\partial^2 f_2}{\partial y\partial x} + \frac{\partial^2 f_3}{\partial z\partial x} - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$= \sum \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$= \sum \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_1}{\partial x}\right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$= \left(\sum \hat{i}\frac{\partial}{\partial x}\right)\left(\overline{\nabla}\cdot\overline{F}\right) - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\sum \hat{i}f_1\right)$$

$$= \overline{\nabla}\left(\overline{\nabla}\cdot\overline{F}\right) - \nabla^2\overline{F}$$

$$= grad(div\overline{F}) - \nabla^2\overline{F}$$

Q.51 The two equal sides of an isosceles triangle are of length a each and the angle θ between them has a probability density function proportional to $\theta(\pi-\theta)$ in the range $\left(0,\frac{\pi}{2}\right)$ and zero otherwise. Find the mean value and the variance of the area of the triangle. **(8)**

Ans.

$$f(\theta) = k\theta(\pi-\theta), o < \theta < \pi/2 \; and \; k\int_0^{\pi/2}\theta(\pi-\theta)\,d\theta = \frac{k\pi^3}{12} = 1 \Rightarrow k = \frac{12}{\pi^3} \therefore f(\theta) = \frac{12}{\pi^3}\theta(\pi-\theta)$$

The area of triangle is $S = \frac{1}{2}a^2\sin\theta \Rightarrow E(S) = \frac{a^2}{2}\int_0^{\pi/2}\sin\theta\frac{12\theta(\pi-\theta)}{\pi^3}d\theta = \frac{12a^2}{\pi^3}$

$$E(S^2) = \frac{3a^4}{\pi^3}\int_0^{\pi/2}\theta(\pi-\theta)\sin^2\theta d\theta = \frac{a^4}{8\pi^2}(3+\pi^2)$$

$$\Rightarrow Var(S) = E(S^2) - (E(S))^2 = a^4\left[\frac{\pi^6+3\pi^4-1152}{8\pi^6}\right]$$

Q.52 Using complex integration, compute $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$ **(8)**

Ans:

The integrand can be written as

$$e^{\cos\theta}\cos(\sin\theta) = \frac{1}{2}e^{\cos\theta}\left[e^{\sin\theta}+e^{-i\sin\theta}\right] = \frac{1}{2}\left[e^z+e^{1/z}\right], \; where \; z = e^{i\theta}$$

$$I = \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = \frac{1}{2}\oint_C\left[e^z+e^{1/z}\right]\frac{dz}{iz} = \oint_C f(z)dz, \; f(z) = \frac{1}{2iz}\left[e^z+e^{1/z}\right], C:|z|=1$$

Now z =0 is an essential singularity of the integrand.

The Laurent series expansion of f(z) is given by

$$f(z) = \frac{1}{2iz}\left[e^z+e^{1/z}\right] = \frac{1}{2iz}\left[\left(1+z+\frac{z^2}{2!}+...\right)+\left(1+\frac{1}{z}+\frac{1}{2!z^2}+...\right)\right], \; \operatorname{Res}_{z=0} f(z) = \frac{1}{i}$$

$$\therefore I = 2\pi i\left[\operatorname{Res}_{z=0} f(z)\right] = 2\pi$$

Q.53 Show that if X has Poisson distribution with mean 1 then its mean deviation about mean is 2/e. **(8)**

Ans.

Poisson distribution is given by P(r) = $\frac{m^r e^{-m}}{r!}$ where m is the mean of the distribution.

Here mean is 1 = m, S.D. = $\sqrt{m} = 1$

$\therefore$ P(r) = $e^{-1}/r! = \frac{1}{e}.\frac{1}{r!}$

We require $E(|r-1|) = \sum_{r=0}^{\infty} P(r)|r-1|$

$= P(0) + \sum_{r=2}^{\infty} P(r)|r-1|$ with $P(0) = \frac{1}{e}$

Mean deviation about mean = $E(|X-1|) = \sum_{x=0}^{\infty} f(x)|X-1| = f(0) + \sum_{2}^{\infty} f(x)(x-1)$

$= f(0) + \sum_{2}^{\infty} xf(x) - \sum_{2}^{\infty} f(x) = 2f(0) - 1 + m = \frac{2}{e}$ (since m =1)

Q.54 A person plays an independent games. The probability of his winning any game is $\frac{a}{a+b}$ (a,b are positive numbers). Show that the probability that the person wins an odd number of games is $\frac{1}{2}\left[(b+a)^m - (b-a)^m\right] / (b+a)^m$ **(8)**

Ans.

The probability that a thing is received by a man is p = a/(a+b), a thing is received by women is q = b/(a+b), hence the probability that (2r+1) things are received by men is $P(X = 2r+1) = \binom{m}{2r+1} p^{2r+1} q^{m-2r-1}, r = 0,1,2...$, the chance that the number of things received by men is odd is $P_0 = \sum P(X = 2r+1) = \binom{m}{1} p^1 q^{m-1} + \binom{m}{3} p^3 q^{m-3} + ...$

$(q+p)^m = q^m + \binom{m}{1} p^1 q^{m-1} + \binom{m}{2} p^2 q^{m-2} + ..., (q-p)^m = q^m - \binom{m}{1} p^1 q^{m-1} + \binom{m}{2} p^2 q^{m-2} - ......$

Subtracting we get $2P_0 = (q+p)^m - (q-p)^m \Rightarrow P_0 = \frac{1}{2}\left[(b+a)^m - (b-a)^m\right]/(b+a)^m$

Q.55 An infinitely long uniform plane plate of breadth π is bounded by two parallel edges and an end right angles to them. This end is maintained at temperature u_0 for all points and the other edge at zero temperature. Determine the temperature at any point of the plate in the steady state. **(8)**

Ans.

In the steady state the temperature u(x, y) at any point P(x, y) satisfies Laplace equation.

Thus, we have to solve the following boundary-value problem: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$

$u(0, y) = 0 = u(\pi, y)$; $u(x,0) = u_0$; $u(x,\infty) = 0$ for $\forall x$.
For solving this we make use of the product solution:
$u(x, y) = X(x)Y(y) \to X''Y + XY'' = 0 \to \frac{X''}{X} = -\frac{Y''}{Y} = -p^2$.
The boundary conditions are $u(0, y) = 0 = u(\pi, y)\ \forall y$,
$u(x,0) = u_0, 0 < x < \pi,\ u(x,\infty) = 0, 0 < x < \pi$ The solution is given as
$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}), u(0, y) = 0 \Rightarrow c_1 = 0$
$\therefore u(x, y) = (c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \because u(\pi, y) = 0 \Rightarrow p = n, also\, u = 0\, as\ y \to \infty, c_3 = 0$
$\therefore u(x, y) = b_n \sin nx e^{-ny}$. Thus $u = \sum_{n=1}^{\infty} b_n \sin nx,\ b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx dx = \begin{cases} 0, & n = even \\ \frac{4u_0}{n\pi}, & n = odd \end{cases}$

Q.56 Using the method of separation of variables solve $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial y}$ **(8)**

Ans.
Given $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial y}$
Let u(x, y) = X(x)Y(y) be the solution of (1), then
$X''Y = 16XY'$ or $\frac{X''}{16X} = \frac{Y'}{Y} = k$
Three possibilities arise
(i) k = 0 (ii) k > 0 (iii) k < 0
For k = 0, X(x) = Ax + B, Y = C; u(x, y) = $C_1 x + C_2$
(ii) k > 0; $X'' - 16kX = 0 \to X = A_1 e^{4\sqrt{x}} + A_2 e^{-4\sqrt{x}}$, $Y(-y) = A_3 e^{ky}$
Hence $u(x, y) = (A_4 e^{4\sqrt{x}} + A_5 e^{-4\sqrt{x}})e^{ky}$.
(iii) $k = -\alpha^2$; $X'' + 16\alpha^2 X = 0 \to X(x) = B_1 \cos 4\alpha x + B_2 \sin 4\alpha x$
and $Y' + \alpha^2 Y = 0 \to Y(y) = B_3 e^{-\alpha^2 y}$
Thus $u(x, y) = (B_4 \cos 4\alpha x + B_5 \sin 4\alpha x)e^{-\alpha^2 y}$.

Q.57 Verify Stoke's theorem for the function $\overline{F} = 2y^3\hat{i} + x^3\hat{j} + z\hat{k}$ where C is the curve of intersection of cone $z = \sqrt{x^2 + y^2}$ by the plane z = 4 and S is surface of cone below z = 4. **(8)**

Ans:
As per Stoke's theorem we have to prove that
$\oint_C \vec{F}.d\vec{r} = \iint_S Curl\vec{F}.\vec{n}dS$
Here $\vec{F} = 2y^3\hat{i} + x^3\hat{j} + z\hat{k}$

$$\text{CurlF} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y^3 & 2x^3 & z \end{vmatrix} = \left(3x^2 - 6y^2\right)\hat{k};$$

$$\hat{n} = \nabla \varphi; \qquad \varphi(x, y, z) = \sqrt{x^2 + y^2} - z$$

$$\varphi_x = \frac{1}{2}\frac{2x}{\sqrt{x^2 + y^2}}, \ \varphi_y = \frac{1}{2}\frac{2y}{\sqrt{x^2 + y^2}}, \ \varphi_z = -1$$

$$n = \nabla \varphi = \frac{x}{y}\hat{i} + \frac{y}{z}\hat{j} - 1.\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} - z\hat{k}}{z}$$

In obtaining $\iint (\nabla \wedge F).\, nds$ we transform it to polar coordinates by using $x = r\cos\theta$, $y = r\sin\theta$. Thus we get

$$\oint_C 2y^3 dx + x^3 dy + zdz = 256\int_{2\pi}^{0}\left[\cos^4\theta - 2\sin^4\theta\right]d\theta = 192\pi$$

Hence Stoke's theorem is verified.

Q.58 Verify Green's theorem for the function $f(x, y) = e^{-x}\sin y,\ g(x, y) = e^{-x}\cos y$ **and** C is the square with vertices $(0,0), (\pi/2, 0), (\pi/2, \pi/2), (0, \pi/2)$. **(8)**

Ans:

$$\oint_C fdx + gdy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} (fdx + gdy),$$

Along $C_1 : y = 0, 0 \le x \le \pi/2, \int_{C_1} e^{-x}(\sin ydx + \cos ydy) = 0$,

$$C_2 : x = \pi/2, 0 \le y \le \pi/2, \int_{C_2} e^{-x}(\sin ydx + \cos ydy) = \int_0^{\pi/2} e^{-\pi/2}\cos ydy = e^{-\pi/2}$$

$$C_3 : y = \pi/2, \pi/2 \le x \le 0, \int_{C_3} e^{-x}(\sin ydx + \cos ydy) = \int_{\pi/2}^{0} e^{-x}dx = e^{-\pi/2} - 1$$

$$C_4 : x = 0, \pi/2 \le y \le 0, \int_{C_4} e^{-x}(\sin ydx + \cos ydy) = \int_{\pi/2}^{0} \cos ydy = -1$$

$\therefore \oint_C fdx + gdy = 2(e^{-\pi/2} - 1)$, Using Green's theorem we get

$$\therefore \oint_C fdx + gdy = \iint_R (-2e^{-x}\cos y)dxdy = \int_0^{\pi/2}\int_0^{\pi/2}(-2e^{-x}\cos y)dxdy = 2(e^{-\pi/2} - 1),$$

Hence Green's theorem is satisfied.

Q.59 Show that the vector field $\bar{F} = 2x\left[y^2 + z^3\right]\hat{i} + 2x^2 y\hat{j} + 3x^2 z^2\hat{k}$ is conservative. Find its scalar potential and work done by it in moving a particle from (-1,2,1) to (2,3,4). **(8)**

Ans:

$$\nabla\times\bar{F}=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x\left[y^2+z^3\right] & 2x^2y & 3x^2z^2 \end{vmatrix}=0$$

Therefore the vector field is conservative.

$$\bar{F}=\nabla\phi\Rightarrow 2x\left[y^2+z^3\right]\hat{i}+2x^2y\hat{j}+3x^2z^2\hat{k}=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow 2x\left[y^2+z^3\right]=\frac{\partial\phi}{\partial x},2x^2y=\frac{\partial\phi}{\partial y},\ 3x^2z^2=\frac{\partial\phi}{\partial z}$$

Thus $\phi=x^2(y^2+z^3)+c\ \therefore W=\int\bar{F}.dr=\phi\Big|_{(-1,2,1)}^{(2,3,4)}=287$

Q.60 Find a normal vector and the equation of tangent plane to surface $z=\sqrt{x^2+y^2}$ at point (3,4,5). **(6)**

Ans:

$f=z-\sqrt{x^2+y^2}$; By definition ∇f is a vector normal to the surface

$$\frac{\partial f}{\partial x}=-\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}},\ \frac{\partial f}{\partial y}=-\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}},\ \frac{\partial f}{\partial y}=1$$

$$\therefore\nabla f=\hat{i}\frac{-x}{y}-\hat{j}\frac{y}{z}+k\ ;\ (\nabla f)_{3,4,5}=\frac{-3}{5}\hat{i}-\frac{4}{5}\hat{j}+\hat{k}$$

Equation of the plane through $(\nabla f)_{3,4,5}$ is

$a(x-3)+b(y-4)+c(z-5)=0$

Here a, b, c are the direction ratios of the normal to the plane and are given by $\frac{-3}{5},\frac{-4}{5},1$.

Using these values we get equation of the tangent plane as $3x+4y-5z=0$.

Q.61 If $\bar{a}$ is a constant vector and $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$ Show that $curl\left(\bar{a}\times\bar{r}\right)=2\bar{a}$ **(5)**

Ans:

Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \nabla\times\left(\bar{a}\times\bar{r}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(\bar{a}\times\bar{r}\right)$

$$=\sum\hat{i}\times\frac{\partial}{\partial x}\left(\bar{a}\times\bar{r}\right)=\sum\hat{i}\times\left(\bar{a}\times\frac{\partial}{\partial x}\bar{r}\right)=\sum\hat{i}\times\left(\bar{a}\times\hat{i}\right)$$

$$=\sum\hat{i}\left\{\left(\hat{i}.\hat{i}\right)\vec{a}-\left(\hat{i}.\vec{a}\right)\hat{i}\right\}=\sum\left\{\vec{a}-a_i\hat{i}\right\}$$

$$=\sum a-\sum a_i\hat{i}=\left(\vec{a}+\vec{a}+\vec{a}\right)-\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$$

$$=3\vec{a}-\vec{a}=2\vec{a}$$

Q.62 Find the values of constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda+2)x,\ 4x^2y + z^3 = 4$ intersect orthogonally at the point (1,-1,2). **(5)**

Ans.
Let $\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x,\ \phi_2 = 4x^2y + z^3 - 4$ the given point (1,-1,2) must lie on both the surfaces. Thus we have $\lambda + 2\mu = (\lambda+2) \Rightarrow \mu = 1$. The two surfaces will intersect orthogonally if normals to them at (1,-1,2) are perpendicular to each other. Therefore at (1,-1,2) $grad\phi_1 \cdot grad\phi_2 = 0 \Rightarrow \lambda = 2.5$

Q.63 Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though CR equations are satisfied at this point. **(8)**

Ans.
If $f(z) = \sqrt{|xy|} = u + iv,\ u = \sqrt{|xy|}, v = 0$ at the origin, we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = 0,$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = 0, \frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = 0, \frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = 0,$$

thus CR equations are satisfied at the origin.

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x(1+im)} = \frac{\sqrt{|m|}}{1+im}$$ which depends on m, thus f(z) is not analytic at the origin.

Q.64 If f(z) = u+*i*v is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$ find f(z) subject to the condition $f\left[\dfrac{\pi}{2}\right] = 0$ **(8)**

Ans.

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y} \Rightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

$$\Rightarrow -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

$$\therefore 2\frac{\partial u}{\partial x} = \frac{(\sin x - \cos x)\cosh y - e^{-y}(\cos x - \cosh y - \sinh y + \sin x) - (\sin x + \cos x)\sinh y + 1}{2(\cos x - \cosh y)^2}$$

$$-2\frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + e^{-y}(\cos x - \cosh y - \sinh y - \sin x) + (\sin x + \cos x)\sinh y + 1}{2(\cos x - \cosh y)^2}$$

Putting x = z, y =0, we get $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{1}{2(1-\cos z)} \Rightarrow f(z) = -\dfrac{1}{2}\cot\dfrac{z}{2} + c$

$$\Rightarrow c = \frac{1}{2} as\ f\left(\frac{\pi}{2}\right) = 0 \Rightarrow f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)$$

Q.65 Discuss the transformation w = z + 1/z and show that it maps the circle |z|=a onto an ellipse. In particular discuss the case when a=1 **(8)**

Ans:

$\frac{dw}{dz} = \frac{(z+1)(z-1)}{z^2}$,it is conformal everywhere except at z=1, -1 which corresponds to w =2,-2 of w plane. Let transform to polar coordinate $w = u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$.....(1),

Eliminating θ we get $\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1$…(2)

Eliminating r we get $\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$ …..(3) From (2) it follows that the circle r = a of z plane are mapped into a family of ellipses in the w plane. The ellipses are confocal since $\left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4$ a constant. In particular, the unit circle r=1 in the z plane gives from (1) $u = 2\cos\theta, v = 0 \Rightarrow |u| \le 2, v = 0$ i.e. the unit circle flattens out to become the segment u=-2 to u=2 of real axis in w plane.

Q.66 Obtain the first three terms of the Laurent series expansion of the function $f(z) = \frac{1}{(e^z - 1)}$ about the point z = 0 valid in the region $0 < |z| < 2\pi$ **(8)**

Ans:

The given function is not analytic when e^z =1, at z = 0 and z= $2\pi ni, n = \pm1, \pm2, ...$
The requires Laurent series expansion is about the point z = 0 . Its region of convergence is $0 < |z| < 2\pi$, we have

$$\frac{1}{e^z - 1} = \frac{1}{z + \left(z^2/2!\right) + \left(z^3/3!\right) + ...} = \frac{1}{z}\left[1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + ...\right)\right]^{-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12}$$

Q.67 Evaluate the integral $\int_0^\infty \frac{x\sin x}{\left(x^2+a^2\right)^2} dx$ by contour integration. **(10)**

Ans:

Since the integrand is an even function we write

$$\int_0^\infty \frac{x\sin x}{\left(x^2+a^2\right)^2} dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{\left(x^2+a^2\right)^2} dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^\infty \frac{xe^{ix}}{\left(x^2+a^2\right)^2} dx$$

Consider the contour integral $I = \int_C f(z)e^{iz}dz, where\, f(z) = \frac{z}{(z^2+a^2)^2}$ The function f(z) is analytic in the upper half plane except for the pole of order 2 at z = ai

$$\operatorname*{Re}_{z=ai} s\left(f(z)e^{iz}\right) = \operatorname*{Lt}_{z\to ai}\frac{d}{dz}\left[\frac{(z-ai)^2 ze^{iz}}{(z^2+a^2)^2}\right] = \frac{e^{-a}}{4a}$$

$$I = \int_{C_R} f(z)e^{iz}dz + \int_{-R}^{R} f(x)e^{ix}dx = \frac{\pi i e^{-a}}{2a}\text{ Now,}$$

$$\left|\int_{C_R}\frac{ze^{iz}}{(z^2+a^2)^2}dz\right| \le \frac{\pi R^2}{(R^2-a^2)^2} \to 0\, as\, R\to\infty$$

$$I = \int_{-\infty}^{\infty} f(x)e^{ix}dx = \frac{\pi i e^{-a}}{2a}\ \therefore \int_0^{\infty}\frac{x\sin x}{\left(x^2+a^2\right)^2}dx = \frac{\pi e^{-a}}{4a}$$

Q.68 Evaluate $\int_C \frac{dz}{z\sin z}$ where C is unit circle described in the positive direction. **(6)**

Ans:
Poles are given by z=0, which is a pole of order 2, sinz =0, z = nπ. Thus only z=0 lies within C. $\operatorname{Re}s\left(z=0\right) = \lim_{z\to 0}\frac{d}{dz}\left(z^2\frac{1}{z\sin z}\right) = 0\ \therefore \int_C \frac{1}{z\sin z}dz = 0$

Q.69 Solve the differential equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod without radiations, subject to the following conditions:

(i) u is not infinite for $t \to \infty$

(ii) $\frac{\partial u}{\partial t} = 0$ for x = 0 and x = L.

(iii) $u = Lx - x^2$, for t = 0 between x = 0 and x = L. **(10)**

Ans:
Let u = X(x)T(t) then $\frac{X''}{X} = \frac{T'}{\alpha^2 T} = -k^2(say) \therefore u = (c_1\cos kx + c_2\sin kx)c_3 e^{-k^2\alpha^2 t}$

From condition (ii) we get

$c_2=0$, kl=nπ. $\therefore u = a_n\cos\frac{n\pi}{l}xe^{-n^2\pi^2\alpha^2 t/l^2}$,

$$\therefore lx - x^2 = a_0 + \sum a_n\cos\frac{n\pi}{l}x,\ a_0 = \frac{l^2}{6}, a_n = \frac{2}{l}\int_0^l (lx-x^2)\cos\frac{n\pi}{l}xdx = -\frac{4l^2}{n^2\pi^2}, n = even$$

$$\therefore u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi}{l}x\right)e^{-\left(\frac{4m^2\pi^2\alpha^2}{l^2}t\right)}$$

Q.70 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the boundary condition

$u(0,y) = \sin y, u \to 0, x\to\infty$ **(6)**

Ans:

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, Let u = X(x)Y(y)

$\frac{X''}{X}=-\frac{Y''}{Y}=p^2 \therefore u=(c_1e^{px}+c_2e^{-px})(c_3\sin py+c_4\cos py)$ using the boundary conditions we get $c_1=0, c_3=1, c_4=0, p=1 \therefore u=e^{-x}\sin y$

Q.71 From a bag containing a black and b white balls, n ball are drawn at random without replacement. Let X denote the number of black balls drawn. Find the probability mass function of random variable X and compute expectation of Y = 2+3X. **(5)**

Ans:

Since $P(X=x)=\frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}, x=0,1,2,....n,\ E(X)=\frac{\sum_{x=0}^{n}x\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}=\frac{na}{a+b}$

$$E(Y)=2+3\left(\frac{na}{a+b}\right).$$

Q.72 If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000 individuals more than two will get a bad reaction. **(5)**

Ans:
Mean= np= 2, P(more than two bad reaction)=1-(P(0)+P(1)+P(2))=1-5/e^2

Q.73 If X is a continuous random variable with p.d.f. given by

$$f(x)=\begin{cases} kx & 0\le x\le 2 \\ 2k & 2\le x\le 4 \\ -kx+6k & 4\le x\le 6 \end{cases}$$

Find the value of k and mean value of X **(6)**

Ans:

By definition $\int_0^6 f(x)dx=1=\int_0^2 kxdx+\int_2^4 2kdx+\int_4^6 (kx+6k)dx$

Or $2k+k-k\left(\frac{x^2}{2}\right)_4^6+6k(x)_4^6=8k\rightarrow k=\frac{1}{8}$

Mean = $\int xf(x)dx=\int_0^2 xkxdx+\int_2^4 x(2k)dx+\int_4^6(-kx^2+6kx)dx$

$$= k\left\{\frac{x^3}{3}\right\}_0^2+2\left(\frac{x^2}{2}\right)_2^4+\left(-\frac{x^3}{3}\right)_4^6+6(x^2)_4^6=\frac{1}{8}\left\{\frac{8}{3}+(16-4)-\frac{1}{3}(216-64)+6(36-16)\right.$$

$$=\frac{1}{8}\left\{\frac{8}{3}+12-\frac{152}{3}+120\right\}=3.$$

Q.74 Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$ **(8)**

Ans:

Let u=X(x)T(t), then $X'T = 2XT' + XT \Rightarrow \frac{X'' - X}{2X} = \frac{T'}{T} = k \Rightarrow X = ce^{(1+2k)x}, T = c'e^{kt}$

thus $u(x,t) = cc'e^{(1+2k)x}e^{kt} \Rightarrow 6e^{-3x} = cc'e^{(1+2k)x} \Rightarrow cc' = 6, k = -2 \therefore u = 6e^{-3x-2t}$

Q.75 If the directional derivative of $\phi = ax^2y + by^2z + cxz^2$ at the point (1,1,1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. Find the value of a,b,c. **(8)**

Ans.

$$\nabla\phi = \left[\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right] = (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)\hat{k} \because |\nabla\phi| = 15$$

Thus we get $(2a+c)^2 + (2b+a)^2 + (2c+b)^2 = 15^2$

But directional derivative is maximum parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \Rightarrow \frac{2a+c}{2} = \frac{2b+a}{-2} = 2c + b \Rightarrow \frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k \Rightarrow k = \pm 5/9$$

Thus we get $a = \pm\frac{20}{9}, b = \pm\frac{55}{9}, c = \pm\frac{50}{9}$

Q.76 If $\overline{A} = \overline{\nabla} \times (\phi\hat{i})$, where $\nabla^2\phi = 0$ show that $\overline{A}.\overline{\nabla} \times \overline{A} = \frac{\partial\phi}{\partial z}.\frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial\phi}{\partial y}.\frac{\partial^2\phi}{\partial z\partial x}$ **(8)**

Ans.

$$A = \nabla_\wedge \varphi\hat{i} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi & 0 & 0 \end{pmatrix} = \hat{j}\varphi_z - \hat{k}\varphi_y$$

$$\nabla_\wedge A = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \varphi_z & -\varphi_y \end{pmatrix} = \hat{j}\varphi_{xy} - \hat{k}\varphi_{xz} - \hat{i}(\varphi_{xx} + \varphi_{yy})$$

$$= \hat{j}\varphi_{xy} + k\varphi_{xz} (\because \nabla^2\varphi = 0)$$

$$\therefore \vec{A}.\nabla_\wedge \vec{A} = (\hat{j}\varphi_z - \hat{k}\varphi_y)(\hat{j}\varphi_{xy} + \hat{k}\varphi_{xz}) = (\varphi_z\varphi_{yx} - \varphi_y\varphi_{zx})$$

Q.77 Show that the integral $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)dx + (yx^2 + 3xy^2)dy$ is independent of the path joining the points (1,2) and (3,4). Hence evaluate the integral. **(8)**

Ans:

For integral to be independent of path $\vec{F} = \nabla\varphi$ obviously

$$\int_{(1,2)}^{(3,4)} \overline{F}.d\overline{r} = \int_{(1,2)}^{(3,4)} \left(xy^2 + y^3\right)dx + \left(yx^2 + 3xy^2\right)dy \text{ and}$$

$$\text{curl}\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 + y^3 & x^2y + 3x^2y & 0 \end{bmatrix} = 0$$

$$\frac{\partial\varphi}{\partial x} = xy^2 + y^3 \rightarrow \varphi(x,y) = \frac{x^2y^2}{z} + y^3x + f(y)$$

$$\frac{\partial\varphi}{\partial y} = 3xy^2 + x^2y \rightarrow \varphi(x,y) = xy^3 + \frac{x^2y^2}{z} + g(x)$$

$$\therefore \int_{1,2}^{3,4}\left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy\right) = \int_{1,2}^{3,4} d\varphi = \varphi\Big]_{1,2}^{3,4} = \frac{9\times16}{2} + (3\times64) - (8) - \frac{1}{2}(4)^2$$

$= 254.$

Q.78 Use Stroke's theorem, to evaluate $\int_C \overline{V}.d\overline{r}$ where $\overline{V} = y^2\hat{i} + xy\hat{j} + zx\hat{k}$ and C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9, z > 0$ oriented in the +ve direction. **(8)**

Ans.

$$\int_C \overline{V}.d\overline{r} = \iint_S (\nabla\times\overline{V}).\hat{n}dS,\ \nabla\times\overline{V} = -z\hat{j} - y\hat{k}, \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3}$$

$$\int_C \overline{V}.d\overline{r} = \iint_S \left(-\frac{2}{3}yz\right)\left(\frac{3}{2}dxdy\right) = -2\int_0^{2\pi}\sin\theta d\theta\int_0^3 r^2dr = 0$$

Q.79 The vector field $\overline{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ is defined over the volume of the cuboid given by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$ Evaluate the surface integral $\iint_S \overline{F}.d\overline{S}$ where S is the surface of the cuboid. **(8)**

Ans:

$$\iint_S (x^2\hat{i} + z\hat{j} + yz\hat{k})dS = \iiint_V div.\overline{F}dV = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(2x + y)dzdxdy = abc\left(a + \frac{b}{2}\right)$$

Q.80 Find the points where CR equations are satisfied for the function $f(z) = xy^2 + ix^2y$. Where does $f'(z)$ exist? Where f(z) analytic? **(8)**

Ans:

Let $u = xy^2, v = x^2 y, \dfrac{\partial u}{\partial x} = y^2, \dfrac{\partial u}{\partial y} = 2xy, \dfrac{\partial v}{\partial x} = 2xy, \dfrac{\partial v}{\partial y} = x^2$

Now $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow y^2 = x^2, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Rightarrow 4xy = 0 \Rightarrow x = y = 0$

Thus at origin C-R equations are satisfied. $f'(z)$ exists at the origin only and f(z) is analytic at the origin only.

Q.81 Find the analytic function $f(z) = u(r,\theta) + iv(r,\theta)$
where $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$ **(8)**

Ans:

We have

$v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2 \Rightarrow \dfrac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta, \dfrac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta$

$\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = -2r\sin 2\theta + \sin\theta, \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} = -2r^2\cos 2\theta + r\cos\theta$ thus

du = d(-r^{2}sin2θ+rsinθ)

$du = \dfrac{\partial u}{\partial r}dr + \dfrac{\partial u}{\partial \theta}d\theta = (-2r\sin 2\theta + \sin\theta)dr + (-2r^2\cos 2\theta - r\cos\theta)d\theta$

$= (2rdr)\sin 2\theta + r^2(2\cos 2\theta d\theta) + (\sin\theta dr + r(\cos\theta d\theta))$

$= -d(r^2\sin 2\theta) + d(r\sin\theta)$

Thus $u = -r^2\sin 2\theta + r\cos\theta + c$

$f(z) = u + iv$ after some simplifications turns out to be

$= ir^2 e^{2i\theta} - ire^{i\theta} + 2i + c$

$= iz^2 + iz + 2i + c$

Q.82 Find the image in w-plane of
(i) the circle with centre (2.5, 0) and radius 0.5
(ii)The interior of the circle in (i) in z plane under the mapping $w = \dfrac{3-z}{z-2}$ **(8)**

Ans:

$w = u + iv = \dfrac{3 - x - iy}{x + iy - 2}$ or $(x + iy - z)(u + iv) = 3 - x - iy$

Equating real and imaginary parts we get

ux – vy -2u = 3 – x, vx – 2v + uy = -y

or (u + 1)x – vy = 2u + 3, vx + (u + 1)y = 2v (1, 2)

or solving eqns(1) & (2) we get

$x = \dfrac{2u^2 + 2v^2 5u + 3}{u^2 + v^2 + 2u + 1}, \; y = \dfrac{-v}{u^2 + v^2 + 2u + 1}$ (3, 4)

Equation of the circle with centre (2.5, 0) and radius 0.5 is $(x - 2.5)^2 + y^2 = 0.25$ (5, 6)

On combining eqns (3,4,5,6) we get

$$\left(\frac{2u^2 + 2v^2 5u + 3}{u^2 + v^2 + 2u + 1} - \frac{5}{2}\right)^2 + \left(\frac{-v}{u^2 + v^2 + 2u + 1}\right)^2 = \frac{1}{4}$$

$(-u^2 - v^2 + 1)^2 + 4v^2 = (u^2 + v^2 + 2u + 1)^2$

After some simplifications we get

$v^2 = (u+1)^2 + (u^2 + v^2 - 1)(u+1)$

Or $u^3 + 2u^2 + u + uv^2 = 0 = u(u^2 + 2u + 1 + v^2) \to u = 0$ which is an equation of imaginary axis.

Equation of the interior of the circle is $\left(x - \frac{5}{2}\right)^2 + y^2 < \frac{1}{4}$

When transformed 2, u, v coordinates we get

$u(u^2 + 2u + 1 + v^2) > 0$ or $u[(u+1)^2 + v^2] > 0$

As $(u+1)^2 + v^2 > 0$ as u > 0 which is an equation of the right half plane.

Q.83 Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent Series valid for $0 < |z+1| < 2$ **(8)**

Ans:

The function $f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{1+z} - \frac{1}{z+1+2}\right]$.

$$= \frac{1}{2(1+z)} - \frac{1}{4}\left[1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - ....\right] = \frac{1}{2(1+z)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16}...$$

Q.84 Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$ **(10)**

Ans:

The integrand can be written as $\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = \int_C f(z) dz$. Poles are z = ± 3*i*, ±*i*

$$\text{Re}\, s(z = i) = \underset{z \to i}{Lt} (z - i) \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} = \frac{1}{16i} - \frac{1}{16},$$

$$\text{Re}\, s(z = 3i) = \underset{z \to 3i}{Lt} (z - 3i) \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} = \frac{7}{48i} + \frac{1}{16}$$

$$\int_C f(z) dz = 2\pi i \left[\frac{1}{16i} + \frac{7}{48i}\right] = \frac{5\pi}{12} \Rightarrow \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$$

Q.85 Use Cauchy Integral formula to evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle |z| = 3 traversed counter clock wise. **(6)**

Ans:

Poles are at z = 1, 2.

Thus

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i \left(\left[\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}\right]_{z=1} + \left[\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}\right]_{z=2}\right) = 4\pi i$$

Q.86 An elastic string of length l which is fastened at its ends x = 0 and x = L is released from its horizontal position (zero initial displacement) with initial velocity g(x) given as $g(x)=\begin{cases} x, & 0 \le x \le L/3 \\ 0, & L/3 < x < L \end{cases}$ Find the displacement of the string at any instant of time.

(10)

Ans:

The equation governing the motion of stretched string is given by

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{--------- (1)}$$

(1) has to be solved under the following initial and b.conditions

b.conditions: u(0, t) = 0 = u(l, t) --------- (2),(3)

initial conditions: u(x, 0) = 0; $\left(\frac{\partial u}{\partial t}\right)_{t=0}=g(x)$ --------- (4),(5)

For solving (1) we assume solution of the form

y(x, t) = X(x)T(t) --------- (6)

Using (6) in (1) we get $XT''=c^2X''T$ --------- (7)

Or $\frac{T''}{c^2T}=\frac{X''}{X}=-k^2$ (on physical ground)

Or $X''+k^2X=0 \rightarrow X(a)=c_1\cos kx+c_2\sin kx$

$T''+k^2c^2T=0 \rightarrow T(t)=c_3\cos kct+c_4\sin kct$

Hence $u(x,t)=(c_1\cos kx+c_2\sin kx)(c_3\cos kct+c_4\sin kct)$ ---- (8)

Using b.condn(2) we get $c_1=0$ and (8) reduces to

$u(x,t)=c_2\sin kx(c_3\cos kct+c_4\sin kct)$

At x = *l*, u = 0 yields $c_2\sin kl=0$ for all

Either $c_2=0$ which gives trivial solution y = 0

Or $\sin kl=0 \rightarrow kl=n\pi$ or $k=\frac{n\pi}{l}$

Hence we get $u_n(x,t)=\sin\frac{n\pi}{l}x\left(c_n\cos\frac{n\pi}{l}ct+a_n\sin\frac{n\pi}{l}ct\right)$

Where $c_2c_3=c_n$, $c_2c_4=a_n$

At t = 0, $u_n(x,t)=0 \rightarrow c_n=0$

Hence $u(x,t)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}ct$

The equation is $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$, $0<x<t$,

With conditions u(0,t)=0, u(L,t)=0, t >0, u(x,0)=0, $\frac{\partial u}{\partial t}(x,0)=g(x)$

Thus $u(x,t)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi c}{l}t\sin\frac{\pi n}{l}x,$

$$a_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{\pi n}{l}xdx = \frac{2}{n\pi c}\left[\frac{\sin n\pi/3}{(n\pi/L)^2} - \frac{L}{3}\frac{\cos n\pi/3}{(n\pi/L)}\right]$$

$$u(x,t) = \frac{2L^2}{\pi^2 c}\sum_{n=1}^{\infty}\left[\frac{\sin n\pi/3}{(n^2\pi)} - \frac{1}{3n}\cos n\pi/3\right]\sin\frac{n\pi c}{l}t\sin\frac{\pi n}{l}x$$

Q.87 Solve by the method of separation of variables $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Ans:
Let z = X(x)Y(y) . Using this in the given d.eqn. we get

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = a \text{ or } X'' - 2X' - aX = 0,\ Y' + aY = 0$$

Aux. eqn. is $m^2 - 2m - a = 0 \rightarrow m = \frac{2 \pm \sqrt{4+4a}}{2}$, $1 \pm \sqrt{1+a}$

Hence $X(x) = c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}$

$Y(y) = e^{-ay}$

$\therefore z = e^{-ay}\left(c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}\right)$

Q.88 The frequency distribution is given as

$f(x) = \begin{cases} x^3, & 0 \le x \le 1 \\ (2-x)^3, & 1 \le x \le 2 \end{cases}$ Calculate Standard deviation and mean deviation about mean. **(5)**

Ans:

Total Frequency $N = \int_0^1 x^3 dx + \int_1^2 (2-x)^3 dx = \frac{1}{2},\ \mu_1' = \frac{1}{N}\left[\int_0^1 x^4 dx + \int_1^2 x(2-x)^3 dx\right] = 1$

$$\mu_2' = \frac{1}{N}\left[\int_0^1 x^5 dx + \int_1^2 x^2(2-x)^3 dx\right] = \frac{16}{15} \Rightarrow \sigma^2 = \mu_2' - (\mu_1')^2 = \frac{1}{5} \Rightarrow \sigma = \frac{1}{\sqrt{5}}$$

$$M.D.about\ mean = \frac{1}{N}\left[\int_0^1 |x-1|x^3 dx + \int_1^2 |x-1|(2-x)^3 dx\right] = \frac{1}{5}.$$

Q.89 Suppose the life in hours of a certain kind of radio tube has p.d.f.

$f(x) = \begin{cases} \frac{100}{x^2} & x \ge 100 \\ 0 & x < 100 \end{cases}$ Find the distribution function. What is the probability that none of the 3 tubes in a given radio set will have to be replaced during the first 150 hours of operation? What is the probability that all three of the original tubes will be replaced during the first 150 hours? **(6)**

Ans:

Distribution function = $F(x)=\int_{100}^{x}\frac{100}{u^2}du=1-\frac{100}{x}$ Probability that a tube will fast for first 150 hours is given by $P(X\le 150)=F(150)=\frac{1}{3}$ Thus the probability that none of the three tubes will have to be replaced during the first 150 hours is $\left(\frac{1}{3}\right)^3=\frac{1}{27}$ The probability that a tube will not last for the first 150 hours is $\frac{2}{3}$ Hence the probability that all three of the original tubes will have to be replaced during the first 150 hours is $\left(\frac{2}{3}\right)^3=\frac{8}{27}$

Q.90 A variate X has p.d.f.

X	-3	6	9
P(x)	1/6	1/2	1/3

Find E(X), E(X^2) and E$(2X+1)^2$. **(5)**

Ans:

$$E(X)=-3.\frac{1}{6}+6.\frac{1}{2}+9.\frac{1}{3}=\frac{11}{2},\ E(X^2)=9.\frac{1}{6}+36.\frac{1}{2}+81.\frac{1}{3}=\frac{93}{2},$$
$$E(2X+1)^2=4E(X^2)+4E(X)+1=209$$

Q.91 Fit a Poisson distribution to the following data which gives the number of calls per square for 400 squares.

No. of calls per square (x)	0	1	2	3	4	5	6	7	8	9	10
No. of squares (f)	103	143	98	42	8	4	2	0	0	0	0

It is given that $e^{-1.32}=0.2674$ **(8)**

Ans:

Mean = $\frac{\sum fx}{\sum f}=\frac{0\times103+1\times143+2\times98+3\times42+4\times8+5\times4+6\times2+7\times0}{103+143+98+42+8+4+2}$

$=\frac{529}{400}=1.32$

No. of calls	0	1	2	3	4	5	6	7	8	9	10
Probability	.2674	.353	.233	.103	.034	.009	.002	.0004	.00006	.000009	.000001
Frequency	107	141	93.2 =93	41	13.52=14	3.57 =4	.78 =1	.15 =0	.24= 0	0	0

Q.92 Find the directional derivative of $\bar{V}^2$ where $\bar{V}=xy^2\hat{i}+y^2z\hat{j}+xz^2\hat{k}$ at the point (2,0,3) in the direction of the outward normal to the sphere $x^2+y^2+z^2=14$ at (3,2,1). **(8)**

Ans:

$\overline{V}^2 = \vec{V}.\vec{V} = x^2y^4 + y^4z^2 + x^2z^4$;

$\nabla V^2 = \hat{i}\{2x(y^4+z^4)\} + \hat{j}\{(x^2+z^2)4y^3\} + \hat{k}\{2zy^4 + 4x^2z^3\}$

$\therefore$ At the pt (2, 0, 3) $\nabla V^2 = 108(3\hat{i} + 4\hat{k})$

$f = x^2 + y^2 + z^2 \qquad (\nabla f)_{3,2,1} = (6\hat{i} + 4\hat{j} + 2\hat{k})$

Thus directional derivative along the normal

$= 108(3\hat{i} + 4\hat{k}).\dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} = \dfrac{1404}{\sqrt{14}}$

Q.93 A Fluid motion is given by $\overline{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$ is the motion is irrotational? If so, find the velocity potential. **(8)**

Ans:

$$\nabla\times\overline{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix} = 0$$ Thus V is a conservative field.

Now

$\overline{V} = \nabla\phi \Rightarrow (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k} = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$

$\Rightarrow y\sin z - \sin x = \dfrac{\partial\phi}{\partial x}, x\sin z + 2yz = \dfrac{\partial\phi}{\partial y}, \; xy\cos z + y^2 = \dfrac{\partial\phi}{\partial z}$

Integrating partially w.r.t. x, we get $\phi = xy\sin z + \cos x + \psi_1(y,z)$

Integrating partially w.r.t. y, we get $\phi = xy\sin z + y^2z + \psi_2(x,z)$

Integrating partially w.r.t. z, we get $\phi = xy\sin z + y^2z + \psi_1(y,x)$

Thus $\phi = xy\sin z + \cos x + y^2z$

Q.94 A vector field is given by $\overline{F} = \sin y\hat{i} + x(1+\cos y)\hat{j}$ Evaluate the line integral $\int_C \overline{F}\cdot d\overline{r}$ where C is a circular path given by $x^2 + y^2 = a^2$. **(8)**

Ans:

$\oint_C F.d\vec{r} = \oint_C \sin y dx + x(1+\cos y)dy$ ----------- (1)

From Green's theorem we know that

$\oint_C (Mdx + Ndy) = \iint_R \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)dxdy$

Using this theorem the line integral (1) is transformed to

$\iint_R [(1+\cos y) - \cos y]dxdy = \iint_R dxdy$.

Using polar coordinates we get $\int_0^{2\pi}\int_0^a r d\theta dr = \pi a^2$.

Q.95 Find $\iint \bar{F}\cdot\hat{n}dS$ where $\bar{F}=(2x+3z)\hat{i}-(xz+y)\hat{j}+(y^2+2z)\hat{k}$ and S is the surface of the sphere having centre (3,-1,2) and radius 3. **(8)**

Ans:

$$div\bar{F}=3,\ \iint \bar{F}\cdot\hat{n}dS=\iiint_V 3dV=3V,\ V=\frac{4}{3}\pi r^3,\ r=3\Rightarrow V=36\pi\ \therefore \iint \bar{F}\cdot\hat{n}dS=108\pi$$

Q.96 Evaluate $\oint_C \bar{F}.d\bar{r}$ where $\bar{F}=y^2\hat{i}+x^2\hat{j}-(x+z)\hat{k}$ and C is boundary of triangle with vertices (0,0,0), (1,0,0) and (1,1,0). **(8)**

Ans

$$\nabla\times\bar{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & x^2 & -(x+z)\end{vmatrix}=\hat{j}+2(x-y)\hat{k}$$

Since z coordinate is zero thus triangle is in xy plane. Thus $\hat{n}=\hat{k}$.

$$\oint_C \bar{F}.d\bar{r}=\iint_S curl\bar{F}\cdot\hat{n}dS=\int_{x=0}^{1}\int_{y=0}^{x}2(x-y)dxdy=\frac{1}{3}$$

Q.97 Show that the function $z|z|$ is not analytic anywhere. **(8)**

Ans:

Let $w=z|z|\Rightarrow u=x\sqrt{(x^2+y^2)},v=y\sqrt{(x^2+y^2)},\frac{\partial u}{\partial x}=\frac{2x^2+y^2}{\sqrt{(x^2+y^2)}},\frac{\partial v}{\partial y}=\frac{2y^2+x^2}{\sqrt{(x^2+y^2)}},$

now

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ when\ y=x,\ \frac{\partial u}{\partial y}=\frac{xy}{\sqrt{(x^2+y^2)}}=\frac{\partial v}{\partial x}$ thus C-R equations are not satisfied anywhere.

Q.98 Show that the function u(x,y)=4xy-3x+2 is harmonic. Construct the corresponding analytic function w = f(z) in terms of complex variables z. **(8)**

Ans

We have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy\Rightarrow v=-2x^2+2y^2-3y+c.$$

Thus $f(z)=4xy-3x+2+i(-2x^2+2y^2-3y)+ic=-2iz^2-3z+2+ic$

Q.99 Find the Taylor series expansion of the function of the complex variable $f(z)=\frac{1}{(z-1)(z-3)}$ about the point z = 4. Find the region of convergence. **(8)**

Ans

If centre of a circle is z=4, then the distance of the singularities z =1 and z = 3 from the centre are 3 and 1. Hence, if a circle is drawn with centre at z = 4, and radius 1, then within the circle $|z-4|=1$, then given function f(z) is analytic hence it can be expanded in Taylor's series within the circle $|z-4|=1$, which is therefore the circle of convergence.

$$f(z)=\frac{1}{(z-1)(z-3)}=\frac{1}{2}\left[\frac{1}{z-3}-\frac{1}{z-1}\right]=\frac{1}{2}\left[1+(z-4)\right]^{-1}-\frac{1}{6}\left[1+\frac{z-4}{3}\right]^{-1}$$

$$=\frac{1}{3}-\frac{4}{9}(z-4)+\frac{13}{27}(z-4)^2+\ldots$$

Q.100 Evaluate $\int_C \frac{z}{z^2+1}dz$ where $\left|z+\frac{1}{z}\right|=2$. **(8)**

Ans:

Poles are

$$z=\pm i,\ \left|z+\frac{1}{z}\right|=2\Rightarrow\left|\frac{z^2+1}{z}\right|=2\Rightarrow\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)+1=4y^2$$

$$\Rightarrow x^2+y^2-1=\pm 2y\Rightarrow x^2+(y\pm 1)^2=2.$$

These are two circles with centre at (0,1) and (0,-1) with radius $\sqrt{2}$.

Thus $\int_C \frac{z}{z^2+1}dz=0$

Q.101 Using complex variable techniques evaluate the real integral

$$\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta$$ **(10)**

Ans:

$$\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta=real\ part\ of\ \frac{1}{2}\int_0^{2\pi}\frac{1-e^{2i\theta}}{5-4\cos\theta}d\theta=real\ part\ of\ \frac{1}{2i}\int_C\frac{z^2-1}{2z^2-5z+2}dz$$

Poles are z = ½, 2. So inside the contour C there is a simple pole at z = ½.

$$\operatorname{Res}\left(z=\frac{1}{2}\right)=\underset{z\to\frac{1}{2}}{Lt}\left(z-\frac{1}{2}\right)\frac{z^2-1}{(2z-1)(z-2)}=\frac{1}{4},\ \ I=real\ part\ of\ \frac{1}{2i}2\pi i\frac{1}{4}=\frac{\pi}{4}$$

Q.102 Determine the poles and residue at each pole of the function f(z)= cotz **(6)**

Ans:

Poles are given by sinz =0, z = nπ. Thus $\operatorname{Res}\left(z=n\pi\right)=\frac{\cos z}{\frac{d}{dz}(\sin z)}\Bigg|_{(z=n\pi)}=1$

Q.103 A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y=a\sin\left(\pi x/l\right)$ from which it is released at time t = 0. Show that the displacement of any point at a distance x from one end at time t is given by

$y(x,t) = a\sin(\pi x/l)\cos(\pi ct/l)$ **(8)**

Ans:

Vibrations of the stretched string are governed by the wave equation (under usual notations)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{----------------- (1)}$$

Since the end points of the string are fixed for all time, therefore the displacement y(x, t) satisfies the following conditions

y(0, t) = y(l, t) = 0 . ----------------- (2), (3)

Further, as the initial transverse velocity of any point of the string is zero one can write

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \text{----------------- (4)}$$

Also, $y(x,0) = a\sin(\prod \frac{x}{l})$ ----------------- (5)

For obtaining solution of (1) under the two boundry conditions (2, 3) and two initial conditions (4, 5) we use the method of product solution and write

Y(x, t) = X(x) T(t) ----------------- (6)

Combining (1) and (6) we get

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -p^2 \qquad \text{(on physical ground)}$$

Thus, $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ ----------------- (7)

On using b.c.(2) we get $c_1 = 0$, and on using the b.c.(3) we get

$$0 = c_2 \sin px(c_3 \cos pct + c_4 \sin pctc) \qquad \text{----------------- (8)}$$

Since (8) is valid for all time therefore $c_3 \sin px = 0$

c_3 cannot be zero as it shall lead to trivial solution. Therefore the only possibility is

$$\sin pl = 0 \rightarrow p = \frac{n\pi}{l} \qquad \text{----------------- (9)}$$

Consequently, solution (8) assumes the following form:

$$y(x,t) = \sin\frac{n\pi}{l}x\left(A_n \cos\frac{n\pi}{l}ct + B_n \sin\frac{n\pi}{l}ct\right)$$

Where $A_n = C_2C_3$, $B_n = C_2C_4$

$$\frac{\partial y}{\partial t} = \sin\frac{n\pi}{l}x\left[\left(-A\sin\frac{n\pi}{l}ct\right)\frac{n\pi}{l} + \left(B\cos\frac{n\pi}{l}ct\right)\frac{n\pi}{l}\right]$$

At t = 0, $\frac{\partial y}{\partial t} = 0 \rightarrow B = 0$

Hence the solution assumes the following form

$$y_n(x,t) = A_n \sin\frac{n\pi}{l}x\cos\frac{n\pi}{l}ct \qquad \text{------------------ (10)}$$

At t = 0, $y_0(x,0) = y(x) = a\sin\frac{\pi x}{l} = A_n \sin\frac{n\pi}{l}x$

Hence n = 1, $A_1 = a$

$$\therefore y(x,t) = a\sin\left(\frac{\pi x}{l}\right)\cos\frac{\pi ct}{l}.$$

Q.104 An infinitely long plate uniform plate is bounded by two parallel edge and an end at right angles to them. The breadth is π; this end is maintained at a temperature u_0 at all points and other edge at zero temperature. Determine the temperature at any point of the plate in the steady-state. **(8)**

Ans:

In the steady state the temperature u(x,y) at any point P(x,y) satisfies the equation

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$ the boundary conditions are $u(0, y) = 0 = u(\pi, y)\ \forall y,$

$u(x,0) = u_0, 0 < x < \pi,\ u(x,\infty) = 0, 0 < x < \pi$

$u(x,t) = X(x)Y(y)$

Or $\frac{X''}{X} = -\frac{Y''}{Y} = -p^2$ (p is a separation constant)

Thus, X = (Acos px + Bsin px); Y(y) = $Ce^{py} + De^{-py}$

Or u(x, y) = $(C_1\cos px + C_2\sin px)(C_3e^{py} + C_4e^{-py})$

Where A, B, C, D are replaced respectively by C_1, C_2, C_3 & C_4.

The solution is given as

$u(x, y) = (c_1\cos px + c_2\sin px)(c_3e^{py} + c_4e^{-py}), u(0, y) = 0 \Rightarrow c_1 = 0$

$\therefore u(x, y) = (c_2\sin px)(c_3e^{py} + c_4e^{-py}) \because u(\pi, y) = 0 \Rightarrow p = n, also\, u = 0\, as\ y \to \infty, c_3 = 0$

$$\therefore u(x, y) = b_n\sin nxe^{-ny}.\ Thus\ u = \sum_{n=1}^{\infty} b_n\sin nx,\ b_n = \frac{2}{\pi}\int_0^{\pi} u_0\sin nx dx = \begin{cases} 0, & n = even \\ \frac{4u_0}{n\pi}, & n = odd \end{cases}$$

Q.105 Show that under the mapping w = 1/z, all circles and straight lines in the z-plane are transformed to circles and straight lines in the w-plane. **(8)**

Ans:

The equation $a(x^2 + y^2) + bx + cy + d = 0$ represents a circle if $a \neq 0$ and a straight line if

a = 0, in the z-plane. Substituting z = x + *i* y, w = u + *i* v, in w = 1/z and comparing the real and imaginary parts, we get

$$x = \frac{u}{u^2 + v^2},\ y = \frac{-v}{u^2 + v^2},\ we\ get\ \frac{a}{u^2 + v^2} + \frac{bu}{u^2 + v^2} - \frac{cv}{u^2 + v^2} + d = 0$$

$$\Rightarrow a + bu - cv + d(u^2 + v^2) = 0. If\ d \neq 0, \left[u + \frac{b}{2d}\right]^2 + \left[v - \frac{c}{2d}\right]^2 = \frac{b^2 + c^2 - 4ad}{4d^2},$$

is the equation of a circle. If d = 0, we get a + bu – cv =0. We observe the following:

(i) A circle ($a \neq 0$) not passing through the origin ($d \neq 0$) in the z-plane, is transformed into a circle not passing through the origin in the w-plane.

(ii) A circle ($a \neq 0$) passing through the origin (d=0) in the z-plane, is transformed into a straight line not passing through the origin in the w-plane.

(iii) A straight line (a=0) not passing through the origin ($d \neq 0$) in the z-plane, is transformed into a circle passing through the origin in the w-plane.

(iv) A straight line (a=0) passing through the origin (d = 0) in the z-plane, is transformed into a straight line passing through the origin in the w-plane.

Q.106 The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men, now 60, at least 7 will leave to be 70? **(8)**

Ans:

The probability that a man aged 60 will live to be 70 = p = 0.65, q = 0.35, n = 10,
Probability that at least 7 will live to 70 = P(7 or 8 or 9 or 10)

$$= {}^{10}C_7 q^3 p^7 + {}^{10}C_8 q^2 p^8 + {}^{10}C_9 q p^9 + p^{10} = 0.5137$$

Q.107 Solve the telephone equation $\frac{\partial^2 E}{\partial x^2} = LC\frac{\partial^2 E}{\partial t^2} + RC\frac{\partial E}{\partial t}$ when $E(0,t) = E_0 \sin qt$, $E = 0\, as\; x \to \infty$ assuming that $\frac{qL}{R}$ is large compared with unity. **(8)**

Ans:

Let $E = Ae^{px}e^{iqt}$ be the solution of the given equation. Substituting in equation, we get $p^2 = -LCq^2(1 - i\,R/qL) \therefore\; p = \pm iq\sqrt{LC}\,(1 - i\,R/Lq)^{1/2} \approx \pm\left[iq\sqrt{LC} + 1/2R\sqrt{C/L}\right]$ $as\,\frac{R}{qL}\,is\,small$. The boundary condition is satisfied when we take –ve sign. Since q can be both –ve as well as +ve, thus the general solution is $E = e^{-ax}\left\{c_1 e^{iqt+ibx} + c_2 e^{-iqt-ibx}\right\},\; a = \frac{R}{2}\sqrt{\frac{C}{L}},\; b = \frac{q}{\sqrt{LC}}$ Using boundary conditions, we get $c_1 = -c_2 = \frac{E_0}{2i}.\; Thus\;\; E(x,t) = E_0 e^{-ax}\sin(qt + bx)$ where $a = \frac{R}{2}\sqrt{\frac{C}{L}},\; b = \frac{q}{\sqrt{LC}}$.

Q.108 Show that the vector field defined by the vector function $\overline{v} = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$ is conservative. **(8)**

Ans:

If the given vector field is conservative, then it can be expressed as the gradient of a scalar function f(x,y,z), therefore, $\nabla f = \left[\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right] = \overline{v} = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$

Comparing, we get $\frac{\partial f}{\partial x} = xy^2z^2,\; \frac{\partial f}{\partial y} = yx^2z^2,\; \frac{\partial f}{\partial z} = zy^2x^2$ integrating the first equation, we obtain $f(x,y,z) = \frac{1}{2}x^2y^2z^2 + g(y,z)$ substituting in the second and third equation we get that g = k = constant. Hence $f(x,y,z) = \frac{1}{2}x^2y^2z^2 + k$

Q.109 Evaluate $\int_C (x+y)dx - x^2dy + (y+z)dz,\; where\,C : x^2 = 4y,\, z = x,\, 0 \le x \le 2$ **(8)**

Ans:

The parametric equation for C is $x = t, y = \frac{t^2}{4}, z = t, 0 \le t \le 2$ therefore

$$\int_C (x+y)dx - x^2 dy + (y+z)dz = \int_0^2 \left[\left(t + \frac{t^2}{4} \right) - t^2 \frac{t}{2} + \left(t + \frac{t^2}{4} \right) \right] dt = \frac{10}{3}$$

Q.110 If $\nabla.\overline{E} = 0,\ \nabla.\overline{H} = 0,\ \nabla \times \overline{E} = -\frac{\partial \overline{H}}{\partial t},\ \nabla \times \overline{H} = \frac{\partial \overline{E}}{\partial t},$

show that vector E and H satisfy the wave equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$ **(8)**

Ans:

Consider $\nabla \times \left(\nabla \times \overline{E} \right) = \nabla \times \left(-\frac{\partial \overline{H}}{\partial t} \right) = -\frac{\partial}{\partial t} \left(\nabla \times \overline{H} \right) = -\frac{\partial^2 \overline{E}}{\partial t^2}$

$\nabla \times \left(\nabla \times \overline{E} \right) = \nabla(\nabla.\overline{E}) - \nabla^2 \overline{E} = -\nabla^2 \overline{E}.\ Thus\ \nabla^2 \overline{E} = \frac{\partial^2 \overline{E}}{\partial t^2}.\ Similarly\ \nabla^2 \overline{H} = \frac{\partial^2 \overline{H}}{\partial t^2}$

Thus vector E and H satisfy the wave equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$.

Q.111 Using the Green's theorem, show that $\oint_C \frac{\partial u}{\partial n} ds = \iint_R \nabla^2 u dxdy,$ where n is the unit vector outward normal to C. **(8)**

Ans:

Let the position vector of a point on C, be $\overline{r}(s) = x(s)\hat{i} + y(s)\hat{j},$ then the tangent vector to C is given by $\overline{T} = \frac{d}{ds}\overline{r}(s) = \frac{d}{ds}x(s)\hat{i} + \frac{d}{ds}y(s)\hat{j},\ \overline{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$, $\overline{n}$ is the unit normal vector . Thus $\oint_C \frac{\partial u}{\partial n} ds = \oint_C \nabla u.\hat{n} ds$ since $\frac{\partial u}{\partial n}$ is the directional derivative of u in the direction of $\overline{n}$ Using Green's theorem, we get

$$\oint_C \frac{\partial u}{\partial n} ds = \oint_C \left(\frac{\partial u}{\partial x}\frac{dy}{ds} - \frac{\partial u}{\partial y}\frac{dx}{ds} \right) ds = \oint_C \left(\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right) = \iint_R \nabla^2 u dxdy.$$

In obtaining the double integral from line integral,
We have used the following form of the Green's theorem

$$\oint_C \left(Mdx + Ndy \right) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy .$$

Q.112 Use the Divergence theorem to evaluate $\iint_S (\overline{v}.\overline{n})dA, where\ \overline{v} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k},$ and S is the boundary of the region bounded by the paraboloid z = $x^2 + y^2$ and the plane z = 4y. **(8)**

Ans:

We have $\iint_S (\bar{v}.\bar{n})dA = \iiint_D \nabla.\bar{v}dV = \iiint_D dV = \int_{y=0}^{4}\int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}}\int_{z=x^2+y^2}^{4y} dzdxdy = 8\pi$

$$\iiint_V dV = \int_{y=0}^{4}\int_{x-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}}\int_{z=x^2+y^2}^{4y} dzdxdy = 2\int_0^4\int_0^{\sqrt{4y-y^2}}\left(4y-y^2-x^2\right)dxdy$$

$$=2\int_0^4\left[\left(4y-y^2\right)\left(4y-y^2\right)^{1/2}-\frac{1}{3}\left(4y-y^2\right)^{3/2}\right]dy = 2\int_0^4\left(4y-y^2\right)^{3/2}\left(1-\frac{1}{3}\right)dy = \frac{4}{3}\int_0^4\left(4y-y^2\right)^{3/2}dy$$

Put $y = 4\sin^2\theta$, $dy = 8\sin\theta\cos\theta d\theta$; $y^{3/2} = 8\sin^3\theta$, $(4-y)^{3/2} = 8\cos^3\theta$,

$\theta = 4$ y=0 $\pi 12\theta = \dfrac{\pi}{2}$ y = 4

$$\frac{4}{3}\int_0^{\pi/2} 8\sin^3\theta 8\cos^3\theta 8\sin\theta\cos\theta d\theta = \frac{4}{3}(512)\times\int_0^{\pi/2}\sin^4\theta\cos^4\theta$$

$$= \frac{4}{3}\times 512\int_0^{\pi/2}\sin^4\theta\left(1+\sin^4\theta-2\sin^2\theta\right) = \frac{4}{3}\times 512\times\int_0^{\pi/2}\left(\sin^4\theta+\sin^8\theta-2\sin^6\theta\right)d\theta$$

$$= \frac{4}{\cancel{3}}\times 512\times\frac{\cancel{3}}{4}.\frac{1}{2}\frac{\pi}{2}\left\{1+\frac{7}{8}.\frac{5}{6}+2\times\frac{5}{6}\right\} = 4\times 512\frac{\pi}{16}\left(\frac{3}{48}\right) = 8\pi$$

Q.113 Show that the function $f(z) = \bar{z}$ is continuous at the point z = 0, but not differentiable at z = 0. **(8)**

Ans:

Let $z = x+iy,\ \bar{z} = x-iy.\ f(z) = x-iy,\ \Delta z = \Delta x + i\Delta y,\ \Delta\bar{z} = \Delta x - i\Delta y$ now $\lim_{z\to 0} f(z) = \lim_{x\to 0, y\to 0}(x-iy) = 0 = f(0)$. Thus the function $f(z) = \bar{z}$ is continuous at the point z = 0. Now at z = 0,

$\lim_{\Delta z\to 0}\dfrac{f(\Delta z)-f(0)}{\Delta z} = \lim_{\Delta z\to 0}\dfrac{\Delta\bar{z}}{\Delta z} = \lim_{\Delta x\to 0, \Delta y\to 0}\left[\dfrac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\right]$ choosing now the path y = mx, we have as $\Delta x\to 0, \Delta y\to 0$ thus $\lim_{\Delta z\to 0}\dfrac{\Delta\bar{z}}{\Delta z} = \lim_{\Delta x\to 0}\left[\dfrac{\Delta x - im\Delta x}{\Delta x + im\Delta x}\right] = \dfrac{1-im}{1+im}$ which depends on the value of m, thus function $f(z) = \bar{z}$ is not differentiable at z =0.

Q.114 Show that the function $v(x,y) = e^x\sin y$ is harmonic. Find its conjugate harmonic function u(x,y) and the corresponding analytic function f(z). **(8)**

Ans:

We have $v(x,y) = e^x\sin y \Rightarrow v_{xx} = e^x\sin y,\ v_{yy} = -e^x\sin y \Rightarrow v_{xx}+v_{yy} = 0$. Thus the function v(x,y) is harmonic. From Cauchy-Riemann equation $u_x = v_y$ we get $u_x = v_y = e^x\cos y$ Integrating w.r.t x, we get $u(x,y) = e^x\cos y + g(y)$ where g(y) is an arbitrary function of y.

Using Cauchy Riemann equation $u_y = -v_x$ we get $-e^x \sin y + g'(y) = -e^x \sin y$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = constt. = c$ Thus $u(x, y) = e^x \cos y + c$

Thus $f(z) = e^x(\cos y + i\sin y) + c = e^z + c$.

Q.115 Evaluate the integral $\int_c (x + y^2 - ixy)dz,\ where\ C : z = z(t) = \begin{cases} t-2i, & 1 \le t \le 2 \\ 2-i(4-t), & 2 \le t \le 3 \end{cases}$ **(8)**

Ans:

The curve C is continuous but not differentiable at z = 2, as

$\frac{dz}{dt} = \begin{cases} 1, & 1 \le t \le 2 \\ i, & 2 \le t \le 3 \end{cases}$ also $\frac{dz}{dt} \ne 0$ for any t. Therefore the curve C is piecewise smooth.

On the interval [1,2], we have z = t – 2*i*, x=t, y = -2, $\frac{dz}{dt} = 1$ and f(z) = (1+2*i*)t +4, On the interval [2,3], we have

z = 2-*i*(4-t), x=2, y =t-4, $\frac{dz}{dt} = i$ and f(z) = 2 + $(t-4)^2$ -2*i*(t -4),

Hence

$$\int_c (x + y^2 - ixy)dz = \int_1^2 f(z)\frac{dz}{dt}dt + \int_2^3 f(z)\frac{dz}{dt}dt$$

$$= \int_1^2 (4 + t + 2it)dt + \int_2^3 \left(2 + (t-4)^2 - 2i(t-4)\right)idt = \frac{5}{2} + \frac{22}{3}i$$

Q.116 Show that the function $f(z) = Ln[z/z-1]$ is analytic in the region $|z| > 1$, obtain the Laurent series expansion about z = 0 valid in the region. **(8)**

Ans:

The function $f(z) = Ln[z/z-1]$ is not analytic when

$\text{Im}[z/z-1] = -\frac{y}{(x-1)^2 + y^2} = 0, \quad \text{Re}[z/z-1] = \frac{x(x-1) + y^2}{(x-1)^2 + y^2} \le 0,$ These conditions are satisfied when y = 0, 0≤x≤1, The given function is analytic in the region $|z| > 1$. In the region $|z| > 1$, consider the function $f(z) = \frac{1}{z} - \frac{1}{z-1}$,

$\int_C f(z)dz = Ln\left[\frac{z}{z-1}\right], C : |z| = r > 1$. $f(z) = \frac{1}{z} - \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-1} = -\frac{1}{z^2} - \frac{1}{z^3} - ...$

Integrating term by term, we obtain the Laurent series expansion as

$Ln\left[\frac{z}{z-1}\right] = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + ... + k,$ where k is a constant of integration, letting $z \to \infty, k = 0$ thus we get $Ln\left[\frac{z}{z-1}\right] = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + ...,\ 1 < |z| < \infty\ or\ |z| > 1.$

Q.117. Prove that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Find a function v that is conjugate harmonic to u and hence the analytic function $f(z) = u + iv$ with

$f(1) = i + 5$. **(7)**

Ans:

$u_x=3x^2-3y^2+6x$, $u_{xx}=6x+6$

$u_y=-6xy-6y$, $u_{yy}= -6x-6$ $u_{xx}+u_{yy}=0$ or u is harmonic

If u is conjugate then u+iu is unalytic and hence CR-equations are satisfied

$u_x=v_y=3x^2-3y^2+6x \Rightarrow v=3x^2y-y^3+6xy+g(x)$

$v_x=6xy+6y+g^1(x)=-u_g=-(-6xy-6y) \Rightarrow g_1(x)=0 \Rightarrow g(x)=C$

$=3x^2y-y^3+6xy+C$, v(1,0)=C,u(1,0) =5, f(1)=5+ic=5+I, c=1

$f(z) \Rightarrow z^3+3y^2+(1+i)$

Q.118 If f(z) is a regular function of z, then prove that $\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. **(7)**

Ans:

$$p=|f(z)|^2=u^2+v^2 \quad \frac{\partial p}{\partial x}=2uu_x+2vv_x \quad \frac{\partial^2 p}{\partial x^2}=2\left[uu_{xx}+u_x^2+vv_{xx}+v_x^2\right]$$

$$similarly\frac{\partial^2 p}{\partial x^2}=2\left[uu_{yy}+u_y^2+vv_{yy}+v_y^2\right]$$

$$\therefore\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]p=2u(u_{xx}+u_{yy})+2v(v_{xx}+v_{yy})+2(u_x^2+u_y^2)+(v_x^2+v_y^2)$$

$$u,\text{v}\ are\ harmonic \Rightarrow u_{xx}+u_{yy}=v_{xx}+v_{yy}=0, also\ u_x=v_{x,}u_y=v_x$$

$$\Rightarrow\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]p=4\left(u_x^2+u_y^2\right)$$

$$\text{since } f^1(z)=u_x+iv_y \Rightarrow |f^1(z)|=u_x^2+v_y^2$$

$$thus\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]p=4|f^1(z)|$$

Q.119. Find the image of the strip $x \ge 0,\ 0 \le y \le \pi$ under the mapping $w = e^z$. **(4)**

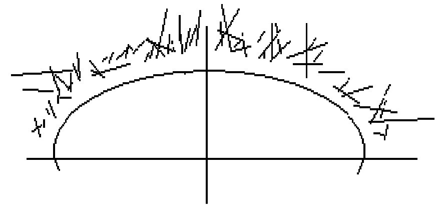
Ans:

$W = Re^{i\phi} = e^{x+iy} \Rightarrow R = e^x, \phi = y$

Image of $0 \le y \le \pi$,$x \ge 0$ is thus $0 \le \phi \le \pi$, 1nR ≥ 0 or R ≥ 1

That is the interior of unit circle$|W|$=1 lying in the upper

half plane as shown in the diagram



Q.120 Find the image of the circle $|z-2| = 3$ under the mapping $w = i(z-2)+3$. **(4)**

Ans:

W=i(z-2)+3, Represent a transtation by 2 to the left followed by rotation through $\frac{\pi}{2}$ followed by transtation by 3 unit to the right. The circle is mapped into a circle of same radius $|W-3|$=3 with center shifting by 1 to the right.

Q.121 Find the linear fractional mapping that maps the points i, 1, 2 + i to 4i, 3-i, ∞ respectively. **(6)**

Ans:

Required mapping is $\frac{w_1 - w}{w_1 - w_2} = \frac{(z_1 - z)}{(z_1 - z_2)}\frac{(z_3 - z_2)}{(z_3 - z)}$

$$\left(4i - w\right)(i-1)(2+i-z) = (i-z)(-3+5i)(1+i)$$

$$4i - w = \left[\frac{z-i}{z-i-2}\right]\frac{(-3+5i)(1+i)}{(i-1)} = \frac{-(z-i-2)4(i+1)-(z-i)(-8+2i)}{(z-i-2)(i-1)}$$

$$= \frac{1-3i+(i-5)z}{(z-i-2)}$$

Q.122. Show that $f(r)\overrightarrow{r}$, where $\overrightarrow{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \left|\overrightarrow{r}\right|$ is irrotational. Find f (r) if it is also solenoidal. **(8)**

Ans:

$$cure\left[f(r)\bar{r}\right] = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{pmatrix}$$

$$= i\left[z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right] + j\left[x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right] + k\left[y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right]$$

$$= \frac{1}{r}f^1(r)\left[i(zy - yz) + j(xz - zx) + K(yx - xy)\right] = 0$$

$f(r)\vec{r}$ is irrotational

$f(r)\vec{r}$ solenoidal if $dwf(r)\vec{r}$=0,or

$$\Delta.\left(f(r)\vec{r}\right) = 3f(r) + x^2\frac{f^1(r)}{r} + y^2\frac{f^1(r)}{r} + z^2\frac{f^1(r)}{r}$$

$$= rf^1(r) + 3f(r) = 0$$

$$= rf^1(r) + 3f(r) = 0 \Rightarrow 1nf(r) = -31nr + k \ or \ f\ (r) = \frac{c}{r^3}, C \text{ a constant}$$

Q.123 The temperature at a point (x, y, z) in a space is given by $T(x, y, z) = x^2 + y^2 - z$. A fly located at the point (4, 4, 2) desires to fly in a direction that gets cooler fastest. Find the direction in which it should fly. Also find the rate decrease of temperature in the direction of flight. **(6)**

Ans:

The direction of maximum decrease is $-\nabla T(x, y, z)$

$= -2x\hat{i} - 2y\hat{j} + \hat{k}$, thus fly should fly in direction

$-8\hat{i} - 8\hat{j} + \hat{k}$, the rate of direction is $\left|8\hat{i} + 8\hat{j} - k\right| = \sqrt{64+64+1} = \sqrt{129}$

Q.124 Evaluate the line integral $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ where C is a simple closed path enclosing origin in its interior. **(7)**

Ans:

Let f(x,y) $= \frac{-y}{x^2+y^2}, g(x,y) = \frac{x}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial g}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \text{ thus } \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$$

except at origin and f,g, $\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ *are* all continuous except at the radius so that so it does not intersect C.

By applying Green's theorem in the region between k and C,

We have $\int_C (fdx + gdy) = \int_k fdx + gdy + \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$

$\int_K fdx + gdy$

Q.125 Show that the following line integral is independent of path C from points $P(-1, 2, 3)$ to $Q(2, 2, 4)$ and hence evaluate the integral

$\int_C (2x\,z + y)dx + (x + z)dy + (x^2 + y)dz$ **(7)**

Ans:

$$Cure\vec{R} = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xg + y & x + z & x^2 + y \end{pmatrix} = i(1-1) + j(2x - 2x) + k(1-1) = 0$$

= line integral is independent of path C.

The integral is then exact defferential

$\frac{\partial \phi}{\partial x} = 2xz + y, \frac{\partial \phi}{\partial x} = x + z, \frac{\partial \phi}{\partial x} = x^2 + y$

integrating first equation w.r.t. X

$\phi = x^2 z + xy + h(y,z) \Rightarrow \frac{\partial \phi}{\partial y} = x + \frac{\partial h}{\partial y} = x + z \Rightarrow \frac{\partial h}{\partial y} = z \Rightarrow h = yz + g(z)$

The $\phi = x^2 y + xy + yz + g(z) \Rightarrow \frac{\partial \phi}{\partial z} = x^2 + y + g^1(z) = x^2 + y \Rightarrow g(z) = C$

Line integral = $\phi(2,2,4) - \phi(-1,2,3) = 28 - 7 = 21$

Q.126 Obtain d'Alembert's solution of the wave equation $\frac{\partial^2 u}{dt^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ with initial conditions u(x, 0) = f (x), $u_t(x,0) = g(x)$. **(11)**

Ans:

Let $v = x + ct, z = x - ct$

then $u_x = u_v v_x + u_z z_x = u_v + u_z$

$u_{xx} = (u_{vv} + u_{zv})v_x + (u_{vz} + u_{zz})z_x = u_{vv} + 2u_{vz} + u_{zz}$

$u_t = u_v v_t + u_z z_t = cu_v + (-c)u_z$

$u_{tt} = c(u_{vv} - u_{zv})v_t + c(u_{vz} - u_{zz})z_t = c^2(u_{vv} - 2u_{zv} + u_{zz})$

The way equation transforms to $u_{vz} = 0$

$\Rightarrow u_v = h(v) \Rightarrow u = \int h(v)dv + \psi(z) = \phi(v) + \psi(z)$

Thus, u(x,t)=$\phi(x+ct) + \psi(x-ct)$, where ϕ, ψ are arbitrary functions gives a solution of wave equation. Applying initial conditions:

u(x,o)=$\phi(x) + \psi(x) = f(x)$

$u_t(x,0) = c\phi^1(x) - c\psi^1(x) = g(x)$

Thus $\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(x)dx + \phi(x_0) = \psi(x_0)$

Hence $\phi(x) = \frac{1}{2}\left[f(x) + \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(x)dx \right]$

$\psi(x) = \frac{1}{2}\left[f(x) - \phi(x_0) + \psi(x_0) - \frac{1}{c}\int_{x_0}^{x} g(x)dx \right]$

The required solution is thus

$$u(x,t) = \phi(x+ct) + \psi(x-ct) = \frac{1}{2}\left[f(x+ct) - f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(x)dx \right]$$

Q.127 A string stretching to infinity in both direction is given the initial displacement $f(x) = \frac{1}{1+8x^2}$ and released from rest. Determine the subsequent motion using d'Alembert's solution obtained in part (a) of the question. **(3)**

Ans:

$f(x) = \frac{1}{1+8x^2}, g(x) = 0$

$$u(x,t)=\frac{1}{2}\left[\frac{1}{1+8(x-ct)^2}\right]+\left[\frac{1}{1+8(x+ct)^2}\right]$$

Q.128 State Cauchy integral formula for derivatives of an analytic function. If $f(a)=\int_C \frac{3z^2+7z+1}{(z-a)}$, where C is the circle $|z|=2$, find $f'(1-i), f''(1-i)$ using Cauchy integral formula. **(7)**

Ans:

Statement of Canchy Integral formula

$$f^1(a)=\int_c \frac{\phi(z)}{(z-a)^2}dz, \phi(z)=3z^2+7z+1$$

$$=\frac{2\overline{\wedge i}}{1!}\phi^1(a), a=1-i \text{ lies inside C}$$

$$=2\overline{\wedge i}\left[6(1-i)+7\right]=2\overline{\wedge}(6+13i)$$

$$f^{11}(a)=2\int_c \frac{\phi(z)}{(z-a)^3}dz=2.\frac{2\overline{\wedge i}}{2!}\phi^{11}(a)=12\overline{\wedge i}$$

Q.129 Identify the singularities of the function $\left(\frac{z^3+z-1}{z(z-1)^2}\right)\sin\frac{1}{z}$. Classify the singularities and find the residues for each of them. **(7)**

Ans:

z=1, a pole of order 2 z=0, essential singularity

$$\text{Res f(z)}=\lim \frac{d}{dz}\left[\frac{z3+z-1}{z}\sin\frac{1}{z}\right]=-\cos 1+3\sin 1$$

$z=1 \qquad z\to 1$

Resf(z)=coeff of $\frac{1}{z}$ in the Laurent expansion in the region $0\langle|z|\langle 1$

$z=0$

$$f(z)=(z_3+z-1)(1-z)^{-2}\frac{1}{z}\sin\frac{1}{z}$$

$$=(z^3+z-1)(1+2z+3z^2+...)[\frac{1}{z^2}-\frac{1}{3!z^4}+\frac{1}{5!z^6}...]$$

$$=\left[-1-z-z^2+z^4+2z^5+3z^6+....\right]\left[\frac{1}{z^2}-\frac{1}{3!z^4}+\frac{1}{5!z^6}\cdots\right]$$

$$\text{Res f(z)}=-1+\frac{2}{5!}+\frac{4}{7!}+\frac{6}{9!}+\cdots\cdots$$

$z=0$

Q.130 State Green's theorem and use this theorem to show that for a solution w (x, y) of Laplace's equation $\nabla^2 w=0$ in a region R with boundary curve C and outer unit normal vector n ,

$$\iint_R \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy = \int_C w \frac{\partial w}{\partial n} ds \qquad \textbf{(7)}$$

Ans:

Statement of Green's theorem

$$\iint_R \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] dxdy = \int_a F_1 dx + F_2 dy$$

Set $F_2 = \dfrac{\partial w^2}{\partial x}, F_1 = \dfrac{\partial w^2}{\partial y}$, then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial}{\partial x}\left[2w\dfrac{\partial w}{\partial x}\right] = 2\left(\dfrac{\partial w}{\partial x}\right)^2 + 2\dfrac{\partial^2 w}{\partial x^2}$

$\dfrac{\partial F_1}{\partial y} = -2\left(\dfrac{\partial w}{\partial y}\right)^2 - 2\dfrac{\partial^2 w}{\partial y^2}$ and $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$

$$= 2\left[\left[\frac{\partial w}{\partial x}\right]^2 + \left[\frac{\partial w}{\partial y}\right]^2 \nabla^2 w\right] = 2\left[\frac{\partial w}{\partial x}\right]^2 + \left[\frac{\partial w}{\partial y}\right]^2$$

$$\therefore \iint_R 2\left[\left[\frac{\partial w}{\partial x}\right]^2 + \left[\frac{\partial w}{\partial y}\right]^2\right] dxdy = \int_c -2w\frac{\partial w}{\partial y}dx + 2w\frac{\partial w}{\partial x}dy$$

$$= \int_c 2w\left[2w\left[-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{\partial y}{\partial s}\right] ds\right]$$

$$\int_c 2w\frac{\partial w}{\partial x} ds$$

Q.131 Verify Stoke's theorem for $\overrightarrow{F} = -y\,\hat{i} + 2\,y\,z\,\hat{j} + y^2\,\hat{k}$, where S is the surface of upper half of the sphere $x^2 + y^2 + z^2 = a^2$, and C is the circular boundary on XOY-plane. **(7)**

Ans:

Cure $\overline{F} = \begin{pmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & 2yz & y^2 \end{pmatrix} = \overline{K}$. Stoke's thm $\int_c \overrightarrow{F}.\overrightarrow{dr} = \iint_s cur\overrightarrow{F}.\overrightarrow{ds}$

RHS: $\iint_s \overline{K}.d\overline{s} = \iint_{x^2+y^2=a^2} dxdy = \overline{\wedge}a^2$, as unit normal to S is $\dfrac{x\hat{i} + y\hat{i} + z\hat{v}}{a}$

LHS: $\int_c -ydx + 2yzdy + y^2dz, C: x^2 + y^2 = a^2, z = 0$

$$= \int_c -ydx = \int_0^{2\overline{\wedge}} a^2 \sin\theta d\theta = \overline{\wedge}a^2$$

RHS = LHS Theorem verified

Q.132 Suppose that two teams A and B are playing a series of games. Team A has a probability p of winning a game against team B. The first team to win three games is declared winner of the series. Find the probability distribution of number of games played in the series for declaring a winner. **(7)**

Ans:

x=no of games in the series $R_x = \{3,4,5\}$

x=3, iff AAA or BBB happens, $P(x=3) = [p^3 + (1-p)^3]$

x=4, iff AABA or BBAB happen, $P(x=4) = \binom{3}{1}p^3(1-p) + \binom{3}{1}p(1-p)^3$

$= 3p(1-p)(p^2+q^2)$

3 cases each $P(x=5) = 1 - P(x=3) - P(x=4) = 6p^2q^2$

x	3	4	5
$p_x(x)$	(p^3+q^3)	$3pq(p^3+q^3)$	$6p^2q^2$

Q.133 The probability density function of a random variable X equals $f(x) = c\,e^{-2x}, x > 0$ and = 0, otherwise. Find c. Also find the probability that X takes a value greater than its expected value. **(7)**

Ans:

$$\int_0^\infty ce^{-2x}dx = 1 \Rightarrow C = 2$$

$$E(x) = \int_0^\infty x.2e^{-2x}dx = 1/2$$

$$P\left[x > \frac{1}{2}\right] = \int_{1/2}^\infty 2e^{-2x}dx = e^{-2\frac{1}{2}} = \frac{1}{e}$$

Q.134 A ticket office can serve 4 customers per minute. The average number of customers arriving to the ticket office for purchase of tickets is 120 per hour. Assuming number of customers arriving to the ticket office follow Poisson distribution, find the probability that ticket office is continuously busy during first 30 minutes of opening. **(6)**

Ans:

X=no arriving in 30 mins. $\lambda = 30 \times \frac{120}{60} = 60$

$$P(x > 120) = 1 - P(x \le 120) = 1 - \sum_{x=0}^{120} \frac{e^{-60}(60)^x}{x^1}$$

Q.135 Sick leaves time X used by employees of a company in one month is roughly normal with mean 1000 hrs and standard deviation 100 hrs. How much time t should be budgeted for sick leaves during next month if t is to be exceeded with a probability of 16%?

Also find the probability that in the next year no more than one month will have sick leave time more than 1200 hrs. (You may use the following values of distribution function $\Phi(z)$ of standard normal distribution $\Phi(.5) = .691$ $\Phi(1.0) = .840$ $\Phi(1.5) = .933$ $\Phi(2.0) = .977$) **(8)**

Ans:

(i) X □ $A(1000,(100)^2, P(X > t) = .16 \quad or\ \Phi\left[\frac{t-1000}{100}\right] = .84 \Rightarrow 1\ or\ t = 1100$

$(ii) P(X > 1200) = 1 - \Phi\left[\frac{1200-1000}{100}\right] = 1 - .9772 = .023$

Required prob = $(.977)^{12} + 12 \times .023 \times (.977)^{11}$

Q.136. Evaluate the integral $\int_{-\infty}^{\infty} \frac{x^2}{\left(x^2+a^2\right)^3} dx$, where a > 0, by using contour integration. **(9)**

Ans:

$Q(z) = \frac{z^2}{\left[z^2+a^2\right]^3}$, has triples poles at z= $\pm$ ai

of these only z=ai is in the upper half plane.

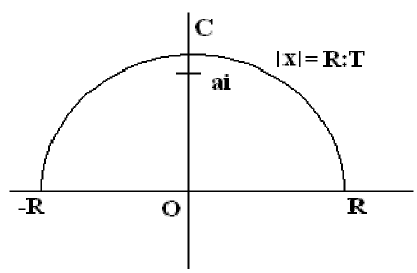
$$\int_C \frac{z^2}{\left[z^2+a^2\right]} = \int_{-R}^{R} Q(x)dx + \int_T Q(z)dz$$

$$|zQ(z)| \to 0 \text{ as } |z| = R \to \infty$$

$$\Rightarrow \int_T Q(z)dz \to 0 \text{ as } R \to \infty$$

$$\therefore \int_{-\infty}^{\infty} Q(x)dx = 2\hat{\pi} i \underset{z=ai}{\text{Residue}} \left[Q(z)\right]$$

$$2\hat{\pi} i \lim_{z \to ai} \frac{1}{21} \frac{d^2}{dz^2} (z-ai)^3 . \frac{z^2}{(z^2+a^2)^3}$$



$$2\bar{\pi} i \left[\frac{2}{(z+ai)^3} - \frac{12z}{(z+ai)^4} + \frac{12z}{(z+ai)^5}\right]_{z=ai}$$

$$= \frac{1}{16a^3 c} \times 2\bar{\pi} i = \frac{\bar{\wedge}}{8a^3}$$

Q.137 Let S be a closed surface of volume V, containing the point P in its interior and let N be the outer unit normal to the surface S at a general point. Show that

$\text{div}\, F(P) = \lim_{V \to 0} \left[\left(\iint_S (F.N) dS\right) \Big/ V\right]$. **(5)**

Ans:

By mean value theorem for triple integrals

$\iiint_v f(x, y, z)dv = f(P_1)v, P_1$ being a point inside V.

By divergence theorem

$$\iiint_v divF \; dv = \iint_s (F.N)ds$$

Thus $divF(P_1) = \dfrac{\iint_s (F.N)ds}{V}$

If V shrinks to P, then $P_1 \to P$

$$V \to 0 \frac{\iint_s (F.N)ds}{V} = divF(P)$$