TYPICAL QUESTIONS & ANSWERS

PART – I

OBJECTIVE TYPES QUESTIONS

Each Question carries 2 marks.

Choose correct or the best alternative in the following:

Q.1 The co-ordinates of the middle points of the sides of a triangle are (4, 2) (3, 3) and (2, 2). Then the co-ordinates of the centroid are

(A) $\left(3, \frac{7}{3}\right)$. **(B)** (3, 3).
(C) (4, 3). **(D)** (4, 7).

Ans: A

Coordinate of the Centroid is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Coordinate of the centroid = $\left(3, \frac{7}{3}\right)$

Q.2 If x, 2x +2, 3x + 3 are first three terms of a G.P. then its 4^{th} term is

(A) 27. **(B)** -27.
(C) 13.5. **(D)** -13.5.

Ans: D

$x(3x+3) = (2x+2)^2$

$3x^2 + 3x = 4x^2 + 8x + 4$

$x^2 + 5x + 4 = 0 \Rightarrow 0 \Rightarrow x = -4, -1$

If x = -1 then three terms are

-1, 0, 0

If x = -4 then the first three terms are

-4, -6, -9

Therefore common ratio is $\frac{3}{2}$

$\therefore$ 4^{th} terms = -13.5

Q.3 The angle made by any diameter of a circle at any point on the circumference is

(A) 90° **(B)** 180°
(C) 45° **(D)** 60°

Ans: A

Q.4 If $^nP_r = 720\ \left(^nC_r\right)$ then the value of r is

(A) 6. **(B)** 5.
(C) 4. **(D)** 7.

Ans:A

$n_{p_r} = 720 n_{c_r}$

Or, $\frac{\angle n}{\angle n - r} = 720 \frac{\angle n}{\angle r - \angle n - r}$

Or, $\angle r = 720 = 1.2.3.4.5.6 = \angle 6$

$\therefore \quad r = 6$

Q.5 $\int \frac{\text{Cos}^2 x - \text{Sin}^2 x}{\text{Sin x Cos x}} dx$ is equal to

(A) $\log(\sin 2x) + C$. **(B)** $\log(\cot 2x) + C$.
(C) $\log(\cos 2x) + C$. **(D)** $\log(\tan 2x) + C$.

Ans: A

$2\int \frac{\cos 2x}{\sin 2x} dx$

$\frac{2 \log \sin 2x}{2} + c$

Q.6 If $20_{Cr} = 20_{Cr-10}$ then 18_{Cr} is equal to

(A) 4896. **(B)** 816.
(C) 1632. **(D)** 408.

Ans: B

$20_{c_r} = 20_{c_{r-10}}$

$\therefore \quad \frac{\lfloor 20}{\lfloor r \lfloor (20-r)} = \frac{\lfloor 20}{\lfloor (r-10) \lfloor (20-r+10)}$

$\lfloor r \lfloor (20-r) = \lfloor (r-10) \lfloor (30-r)$

$\therefore \quad r = 15$

$\therefore \quad 18_{c_5} = \frac{18}{\lfloor 15 \lfloor 3}$

$\frac{18 \times 17 \times 16}{6} = 816$

Q.7 $\lim_{x \to 5} \frac{x^4 - 625}{x^3 - 125}$ is

(A) $\frac{20}{3}$. **(B)** 5.
(C) Not defined. **(D)** $\frac{4}{3}$.

Ans: A

$$\lim_{x\to 5}\frac{(x^2)^2-(y^2)^2}{x^3-5^3}$$

$$= \lim_{x\to 5}\frac{(x-5)(x+5)(x^2+25)}{(x-5)(x^2+5x+25)}$$

$$= \frac{500}{75}=\frac{20}{3}$$

Q.8 $\dfrac{1-\tan^2 165^\circ}{1+\tan^2 165^\circ}$ is equal to

(A) $\frac{1}{2}$. **(B)** $-\frac{\sqrt{3}}{2}$.

(C) $-\frac{1}{2}$. **(D)** $\frac{\sqrt{3}}{2}$.

Ans: D

$$\frac{1-\tan^2 165^0}{1+\tan^2 165^0}=\cos 330^0=\cos 30^0=\frac{\sqrt{3}}{2}$$

Q.9 The equation of the straight line which makes equal intercepts on the axes and passes through the point (1, 2) is

(A) $x + y = 3$ **(B)** $x + 2y = 5$

(C) $x - y = 1$ **(D)** $2x + y = 4$

Ans: A

Straight line having equal intercepts on axes is $x + y = a$. If it passes through (1, 2), then $a = 3$. Hence required straight line $x + y = 3$.

Q.10 Area of the triangle whose vertices are (a, b) (a, a + b), (-a, -a + b) is

(A) a^2b^2 **(B)** $a^2 + b^2$

(C) a^2 **(D)** b^2

Ans: C

Area of reqd. $\Delta = \frac{1}{2}\left[x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1\right]$

$= a^2$

Q.11 $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$ is

(A) 1 **(B)** $\frac{1}{2}$

(C) $\frac{1}{4}$ **(D)** Zero

Ans: B

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-(1-2\sin^2 x/2)}{x^2}$$
$$= \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2} = \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin x/2}{x/2}\right)^2 = \frac{1}{2}$$

Q.12 The point on the curve $y^2 = 4x$ at which the tangent to the curve is parallel to y = x is

(A) (0, 0) **(B)** $(2,\ 2\sqrt{2})$
(C) (4, 4) **(D)** (1, 2)

Ans: D

Here $\frac{dy}{dx} = \frac{4}{2y} = \frac{1}{\sqrt{x}}$. If tangent is parallel to y = x, $\frac{dy}{dx} = \frac{1}{\sqrt{x}} = 1$

$\therefore \quad x = 1,\ y = 2$

Q.13 $\int \frac{\text{Sin}^3 x - \text{Cos}^3 x}{\text{Sin}^2 x \,\text{Cos}^2 x}\, dx$ is equal to

(A) tan x – cot x **(B)** tan x + cot x
(C) sec x + cosec x **(D)** sec x - cosec x

Ans: C

$$\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx = \int (\tan x \sec x - \cos ecx \cot x) dx = \sec x + \cos ecx$$

Q.14 $\int_0^{\frac{\pi}{2}} \text{Sin}^3 x\, dx$ is equal to

(A) $\frac{2}{3}$ **(B)** $\frac{3}{2}$
(C) $\frac{\pi}{2}$ **(D)** $\frac{\pi}{4}$

Ans: A

$$\int_0^{\pi/2} \sin^3 x dx = \frac{2}{3} \quad \text{(By formula)}$$

Q.15 Solution of differential equation $\frac{dy}{dx} = e^{x-y}$ is

(A) $e^x + e^y$ = const **(B)** $e^x - e^y$ = const
(C) $e^x \,.\, e^y$ = const **(D)** e^x / e^y = const

Ans: $\frac{dy}{dx} = e^x e^{-y} \therefore e^x dx = e^y dy \; or \; e^x = e^y + const \; or \; e^x - e^y = const$

Q.16 Period of Sin (2x + 3) is

(A) 2π **(B)** $\frac{3\pi}{2}$

(C) π **(D)** $\frac{\pi}{2}$

Ans: C

$\sin(2x+3)=\sin(2x+3+\pi)=\sin[2(x+\pi)+3]$

$=\sin[2(x+2\pi)+3]=$ ………Hence period is π.

Q.17 The value of Sin 105^0 + Cos 105^0 is

(A) $\frac{\sqrt{3}}{2}$ **(B)** $\frac{1}{\sqrt{3}}$

(C) $\frac{1}{2}$ **(D)** $\frac{1}{\sqrt{2}}$

Ans: D

$\sin 105^0+\cos 105^0=\sin(60^0+45^0)+\cos(60^0+45^0)=\sin 60^0$

$\cos 45^0+\cos 60^0\sin 45^0+\cos 60^0\cos 45^0-\sin 60^0\sin 45^0$

$=\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}+\frac{1}{2}\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}$

Q.18 If pth, $(2p)^{th}$ and $(3p)^{th}$ terms of a G.P. are x, y, z respectively, then x, y, z are in

(A) A.P. **(B)** H.P.

(C) G.P. **(D)** None of these

Ans: C

If a, ar, ar^2, ar^3, ……….. be the G.P. then T_p = x = ar^{p-1}, T_{2p} = y ar^{2p-1}, T_{3p} = Z = ar^{3p-1}. Evidently y^2 = xz. Hence x, y, z are in G.P.

Q.19 Sum of the series $S=1^2-2^2+3^2-4^2+...-100^2+101^2$ is equal to

(A) 348551 **(B)** -1000

(C) 5151 **(D)** None of the above

Ans: C

Q.20 The value of tan 15° is

(A) $2-\sqrt{3}$ **(B)** $-2+\sqrt{3}$

(C) $2+\sqrt{3}$ **(D)** $-2-\sqrt{3}$

Ans: A

Q.21 In a triangle ABC, let a = BC, b = CA and c = AB. If $\angle B=60^\circ$, then

(A) $(a-b)^2=c^2-ab$ **(B)** $(b-c)^2=a^2-bc$

(C) $(c-a)^2=b^2-ac$ **(D)** None of the above

Ans: C

Q.22 The circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ cut orthogonally if the value of p is

(A) 3 **(B)** -2
(C) -3 **(D)** 1

Ans: A

Q.23 The eccentricity of the ellipse $16x^2 + 25y^2 = 400$ is

(A) 3 **(B)** $5/3$
(C) 5 **(D)** $3/5$

Ans: D

Q.24 The derivative of $-\cos(\log x)$ is

(A) $\sin(\log x)$ **(B)** $\frac{\sin(\log x)}{x}$
(C) $-\sin(\log x)$ **(D)** $-\cos\left(\frac{1}{x}\right)$

Ans: **B**

Q.25 The value of the $\lim_{x\to 0} \frac{\sin xe^{-x}}{x}$ is

(A) 0 **(B)** 1
(C) e **(D)** Does not exist

Ans: B

Q.26 The integral $\int_0^1 xe^x$ is equal to

(A) $e - 1$ **(B)** $e + 1$
(C) 0 **(D)** 1

Ans: D

Q.27 The area under the curve $y = x^2$ between x = 0 and x = 1 is

(A) 1 **(B)** $1/2$
(C) $1/3$ **(D)** $1/4$

Ans: C

Q.28 The solution of $\frac{dy}{dx} = y^2, y(1) = -1$ is

(A) $y = -\frac{1}{x}$ **(B)** $y = \frac{1}{x}$

(C) $y = x + 1$ **(D)** $y = x^2 + 1$

Ans: A

Q.29 If one root of the equation $2x^2 - 10x + K = 0$ is $\frac{2}{3}$ of the other root, then K is

(A) 2 **(B)** 8
(C) 10 **(D)** 12

Ans: D.

If one root = α, the other root = $\frac{2}{3}\alpha$

$\therefore$ Sum of roots = $\alpha + \frac{2}{3}\alpha = \frac{10}{2}$ or $\alpha = 3$

$\therefore$ Roots are 3 and 2 and product of roots = $3 \times 2 = \frac{k}{2}$

$\therefore$ k = 12.

Q.30 The centroid of the triangle formed by the straight lines $y + x = 3, y - x = 3, y = 0$ is

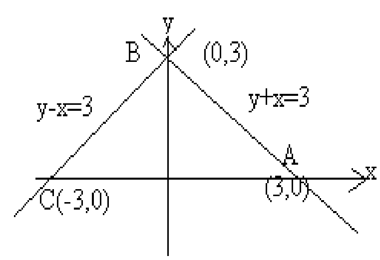
(A) (0, 0) **(B)** (1, 0)
(C) (0, 1) **(D)** (1, 1)

Ans: C.

The points of intersection of the

Given straight lines are A(3, 0), B(0, 3), C(-3, 0)

$\therefore$ centroid of Λ ABC is $\left(\frac{3+0-3}{3}, \frac{0+3+0}{3}\right)$ *or* (0, 1)



Q.31 The distance between the parallel lines 3x + 4y + 5 = 0 and 3x + 4y + 15 = 0 is

(A) 1 **(B)** 2
(C) 3 **(D)** 5

Ans: B.

The distance of origin from the line 3x + 4y +5 = 0 is

$$p_1 = \frac{3\times 0 + 1\times 0 + 5}{\sqrt{3^2 + 4^2}} = 1$$

The distance of origin from the line 3x + 4y +15 = 0 is

$$p_2 = \frac{3\times 0+1\times 0+15}{\sqrt{3^2+4^2}} = 3$$

$\therefore$ distance between parallel lines = $p_2 - p_1 = 3-1 = 2.$

Q.32 $\lim\limits_{x\to 0} \frac{\sin mx - \sin nx}{x}$, where m $\neq$ n is equal to

(A) m **(B)** n
(C) m – n **(D)** m + n

Ans: C.

$$\lim_{x\to 0}\frac{\sin mx - \sin nx}{x} = \lim_{x\to 0} m\frac{\sin mx}{mx} - \lim_{x\to 0} n\frac{\sin nx}{nx} = m-n$$

Q.33 If $y = \sin^2 2x,$ then $\frac{dy}{dx}$ is equal to

(A) 2 sin 4x **(B)** 4 sin 2x
(C) sin 4x **(D)** 2 sin 2x

Ans: A.

$$y=\sin^2 2x, put 2x = t \therefore \frac{dy}{dx} = \frac{dy}{dt} = .\frac{dt}{dx}.$$

$$\therefore y = \sin^2 t \quad \therefore \frac{dy}{dt} = 2\sin t\cos t, \frac{dt}{dx} = 2,$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = (2\sin t\cos t)(2) = 2\sin 2t = 2\sin 4x.$$

Q.34 $\int \frac{dx}{1+\sin x}$ is equal to

(A) $\sin\frac{x}{2}+\cos\frac{x}{2}$ **(B)** $\log\left(1+\sin x\right)$
(C) tan x + sec x **(D)** tan x – sec x

Ans: D.

$$\int\frac{dx}{1+\sin x} = \int\frac{1-\sin x}{(1-\sin x)(1+\sin x)}dx = \int\frac{1-\sin x}{(1-\sin^2 x)}dx = \int\frac{1-\sin x}{(\cos^2 x)}dx$$

$$= \int\frac{1}{\cos^2 x}dx - \int\frac{\sin x}{\cos x}\frac{1}{\cos x}dx = \int\sec^2 x dx - \int\tan x\sec x dx = \tan x - \sec x$$

Q.35 $\int\limits_0^\infty \frac{e^x}{1+e^{2x}}dx$ is equal to

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{4}$

(C) 1 **(D)** 0

Ans: B.

$$\int_0^\infty \frac{e^x}{1+e^{2x}}dx, \quad \text{put } e^x = t \quad \therefore e^x dx = dt$$

$$= \int_1^\infty \frac{dt}{1+t^2} = |\tan^{-1} t|_1^\alpha = \tan^{-1}\infty - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Q.36 The solution of the differential equation $\sqrt{y}\,dx + \sqrt{x}\,dy = 0$ is

(A) $\sqrt{x} + \sqrt{y} = \text{const}$ **(B)** $\sqrt{xy} = \text{const}$

(C) $x\sqrt{y} + y\sqrt{x} = \text{const}$ **(D)** $\sqrt{x}/\sqrt{y} = \text{const}$

Ans: A.

$$\sqrt{y}dx + \sqrt{x}dy = 0 \quad or \quad \frac{dx}{\sqrt{x}} + \frac{dx}{\sqrt{y}} = 0 \quad or\ \sqrt{x} + \sqrt{y} = const.$$

Q.37 The value of $\sin 75^\circ - \cos 75^\circ$ is equal to

(A) 1 **(B)** $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ **(D)** zero

Ans: C.

$$\sin 75^\circ - \cos 75^\circ = \sin(45^\circ + 30^\circ) - \cos(45^\circ + 30^\circ)$$

$$= \sin 45 \ \cos 30 \ + \cos 45 \ \sin 30 \ - \cos 45 \ \cos 30 \ + \sin 45 \ \sin 30$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} - \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{1}{\sqrt{2}}$$

Q.38 The value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ is

(A) 2π **(B)** π

(C) $\frac{\pi}{2}$ **(D)** $\frac{\pi}{4}$

Ans: D.

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}+\frac{1}{3}} = \tan^{-1}\frac{5/6}{6-1/6} = \tan^{-1}1 = \frac{\pi}{4}$$

Q.39 If $n_{c_{12}} = n_{c_8}$, then n is equal to

(A) 8 **(B)** 12
(C) 16 **(D)** 20

Ans: D

$n_{c_{12}} = n_{c_8}$

$$\frac{\lfloor n}{\lfloor 12 \lfloor n-12} = \frac{\lfloor n}{\lfloor 8 \lfloor n-8}$$

Or $\dfrac{\lfloor n-8}{\lfloor n-12} = \dfrac{\lfloor 12}{\lfloor 8}$

(n - 8)(n - 9)(n - 10)(n - 11) = 12.11.10.9 = (20 - 8)(20 - 9)(20 - 10)(20 - 11)
$\therefore n = 20$

Q.40 $\lim\limits_{x \to 0} \dfrac{2\sin x - \sin 2x}{x^3}$ is equal to

(A) 0 **(B)** 1
(C) 2 **(D)** 3

Ans: B

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = \lim_{x \to 0} \frac{2\sin x(1-\cos x)}{x^3}$$

$$= \lim_{x \to 0} \frac{2\sin x \, 2\sin^2 \frac{x}{2}}{x^3} = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

= 1 x 1 = 1

Q.41 If the point P(x, y) is equidistant from the points $A(a+b, b-a)$ and $B(a-b, a+b)$, then

(A) bx = ay **(B)** ax = by
(C) x = y **(D)** x + y = 0

Ans: A

$$(x-a-b)^2 + (y-b+a)^2 = (x-a+b)^2 + (y-a+b)^2$$

$$\Rightarrow x^2 + a^2 + b^2 - 2ax - 2bx + 2ab + y^2 - 2by + 2ay - 2ab + a^2 + b^2$$

$$= x^2 + a^2 + b^2 + 2xb - 2xa - 2ab + y^2 + a^2 + b^2 - 2ya - 2yb + 2ab$$

$$\Rightarrow xb = ya$$

Q.42 The area of the triangle formed by the lines y = a + x, y = a – x, y = 0, where a > 0, is

(A) 1 **(B)** a
(C) a^2 **(D)** zero

Ans: C

y = a + x $\Rightarrow y - x = a$
y = a – x $\Rightarrow y + x = a$

Solving, we get y = a, x = 0
Therefore vertex is (0, a) another vertices are (-a, 0), (a, 0).

$\therefore$ Area of the triangle is $\frac{1}{2}[0(0-0)-a(0-a)+a(a-0)]$

$$= \frac{1}{2}.2a^2 = a^2$$

Q.43 If $\frac{x}{y}+\frac{y}{x}=2$ and $Y \neq x$, then $\frac{dy}{dx}$ is equal to

(A) 1 **(B)** 2
(C) -1 **(D)** -2

Ans: A

$$\frac{x}{y}+\frac{y}{x}=2$$
$$\Rightarrow x^2+y^2=2xy$$
$$\Rightarrow (x-y)^2=0$$
$$\Rightarrow 2(x-y)\left(1-\frac{dy}{dx}\right)=0$$
$\because y \neq x$ thus
$$\frac{dy}{dx}=1$$

Q.44 $\int \frac{\text{Cos}2x}{\text{Cos}^2x\,\text{Sin}^2x}dx$ is equal to

(A) sec x + cosec x **(B)** $\sec x - \operatorname{cosec} x$
(C) $-\sec x\ \operatorname{cosec} x$ **(D)** $\sec x\ \operatorname{cosec} x$

Ans: C

$$\int \frac{\cos 2x}{\cos^2 x \sin^2 x}dx$$
$$= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x}dx$$
$$= -\cot x - \tan x$$
$$= -\sec x.\operatorname{cosec} x$$

Q.45 The area bounded by the parabola $y^2 = 4ax$ and its latus rectum is

(A) a^2 **(B)** $\frac{2}{3}a^2$
(C) $\frac{4}{3}a^2$ **(D)** $\frac{8}{3}a^2$

Ans: D

The parabola is symmetrical about the x – axis

Area = $2\int_0^a y\,dx$

$A = 2\int_0^a 2\sqrt{ax}\,dx$

$= 4\sqrt{a}\int_0^a \sqrt{x}\,dx$

$= \frac{8}{3}\sqrt{a}.a^{3/2} = \frac{8}{3}a^2$

Q.46 The solution of differential equation $\frac{dy}{dx} = e^{x-y} + 2xe^{-y}$ is

(A) $y = xe^{x-y} + x^{z-y}e + c$ **(B)** $e^y = e^x + x^2 + c$

(C) $y = e^{x-y} + x^{z-y}e + c$ **(D)** $e^{-y} = e^x + 2x + c$

Ans: B

$\frac{dy}{dx} = e^{x-y} + 2xe^{-y}$

$\frac{dy}{dx} = e^x.e^{-y} + 2xe^{-y}$

$\Rightarrow e^y dy = (e^x + 2x)dx$

$= e^y = e^x + x^2 + c$

Q.47 Value of $\sin^{-1} x + \cos^{-1} x$ is

(A) 2π **(B)** π

(C) $\frac{\pi}{2}$ **(D)** $\frac{\pi}{4}$

Ans: C

$\sin^{-1} x + \cos^{-1} x$

$= \frac{\pi}{2}$

Q.48 Value of $(\sin 3A - \sin A)\cos A - (\cos 3A + \cos A)\sin A$ is

(A) 0 **(B)** 1

(C) $\frac{1}{2}$ **(D)** $\frac{1}{\sqrt{2}}$

Ans: A

$\sin 3A \cos A - \sin A \cos A - \cos 3A \sin A - \cos A \sin A$

$= \sin(3A - A) - 2\sin A \cos A$

$= \sin 2A - \sin 2A = 0$

Q.49 The number of terms in the sequence $\frac{5}{2}, 5, 10, \ldots\ldots\ldots\ldots, 640$ are

(A) 8 **(B)** 9
(C) 10 **(D)** 6

Ans: B

$a = \frac{5}{2}$ C.R = 2 $t_n = 640$

$t_n = a.r^{n-1} \quad \Rightarrow 640 = \frac{5}{2}.2^{n-1}$

$128 = \frac{1}{2}.2^{n-1} = 2^{n-2}$

$2^7 = 2^{n-2} \quad \therefore n = 9$

Q.50 First three terms in the expansion of $\left(1-2x^3\right)^{11/2}$ are

(A) $1+11x^3+\frac{99}{2}x^6+\ldots\ldots\ldots\ldots\ldots\ldots$ **(B)** $1+\frac{11}{2}x^3+99x^6+\ldots\ldots\ldots\ldots\ldots$

(C) $1-\frac{11}{2}x^3-\frac{99}{2}x^6+\ldots\ldots\ldots\ldots\ldots$ **(D)** $1-11x^3+\frac{99}{2}x^6+\ldots\ldots\ldots\ldots\ldots$

Ans: D

$$(1-x)^n = 1-nx+\frac{n(n-1)}{2}x^2+\ldots\ldots\ldots$$

$$\therefore (1-2x^3)^{\frac{11}{2}} = 1-\frac{11}{2}.2x^3+\frac{\frac{11}{2}\left(\frac{11}{2}-1\right)}{2}(2x^3)^2+\ldots\ldots\ldots\ldots$$

$$= 1-11x^3+\frac{99}{2}x^6+\ldots\ldots\ldots$$

Q.51 Value of $\tan 105^\circ$ is

(A) $-\left(2+\sqrt{3}\right)$ **(B)** $2-\sqrt{3}$
(C) $-2+\sqrt{3}$ **(D)** $2+\sqrt{3}$

Ans: A

$$\tan 105^0 = \tan(60^0+45^0) = \frac{\tan 60^0 + \tan 45^0}{1-\tan 60^0 \tan 45^0}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{(\sqrt{3}+1)^2}{3-1} = \frac{-(3+1+2\sqrt{3})}{2}$$

$$= -(2+\sqrt{3})$$

Q.52 If $\cos A = \frac{4}{5}$, then the value of cos 2A is

(A) $\frac{3}{25}$ **(B)** $\frac{1}{25}$

(C) $\frac{2}{25}$ **(D)** $\frac{7}{25}$

Ans: D

$$\cos 2a = 2\cos^2 a - 1 = 2\times\frac{16}{25} - 1 = \frac{32}{25} - 1 = \frac{7}{25}$$

Q.53 The value of 'x' such that PQ = QR, where P, Q and R are (6, -1), (1, 3) and (x, 8) respectively is given by

(A) 5, –3 **(B)** 3, 5

(C) 2, 5 **(D)** 2, 3

Ans: A

$$PQ = \sqrt{(6-1)^2 + (-1-3)^2} = \sqrt{25+16} = \sqrt{41}$$

$$QR = \sqrt{(1-x)^2 + (3-8)^2} = \sqrt{1-2x+x^2+25}$$

Or $41 = x^2 - 2x + 26$

Or $x^2 - 2x - 15 = 0$

Or $x^2 - 5x + 3x - 15 = 0 \qquad \Rightarrow x = 5, -3$

Q.54 Slope of the line passing through the points $\left(\frac{5}{2}, 3\right)$ & $\left(0, \frac{3}{4}\right)$ is

(A) $\frac{9}{10}$ **(B)** $\frac{3}{5}$

(C) $\frac{9}{5}$ **(D)** $\frac{10}{9}$

Ans: A

$$\tan\theta = \frac{y_1 - y_2}{x_1 - x_2} = \frac{9}{10}$$

Q.55 $\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3}$ is equal to

(A) $\frac{1}{3}$ **(B)** $\frac{2}{3}$

(C) $\frac{1}{2}$ **(D)** $-\frac{1}{3}$

Ans: C

$$\lim_{x \to 3} \frac{(x-3)(x-1)}{(x-3)(x+1)} = \frac{1}{2}$$

Q.56 If $y = \sin^{-1}\left(3x - 4x^3\right)$ then $\frac{dy}{dx}$ is equal to

(A) $\frac{3}{\sqrt{1-x^2}}$ **(B)** $\frac{-3}{\sqrt{1-x^2}}$

(C) $\frac{2}{1+x^2}$ **(D)** $\frac{-1}{\sqrt{1-x^2}}$

Ans: A

$$\frac{dy}{dx} = \frac{3(1-4x^2)}{\sqrt{1-(3x-4x^3)^2}}$$

$$= \frac{3(1-4x^2)}{(1-4x^2)\sqrt{1-x^2}} = \frac{3}{\sqrt{1-x^2}}$$

Q.57 $\int \sin 3x \sin 2x \, dx$ is equal to

(A) $\frac{1}{2}\left(\cos x + \frac{\cos 5x}{5}\right)$ **(B)** $\frac{-1}{2}\left(\cos x + \frac{\sin 5x}{5}\right)$

(C) $\frac{1}{2}\left(\sin x - \frac{\sin 5x}{5}\right)$ **(D)** $\frac{1}{2}\left(\sin x + \frac{\sin 5x}{5}\right)$

Ans: C

$$\int \sin 3x \sin 2x dx$$

$$= \frac{1}{2}\int (\cos x - \cos 5x) dx$$

$$= \frac{1}{2}\left(\sin x - \frac{\sin 5x}{5}\right)$$

Q.58 Order and degree of the differential equation $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + 4y = \sin x$ is given by

(A) 3, 2 **(B)** 2, 3

(C) 1, 3 **(D)** 3, 1

Ans: D

Order – 3 [Power of higher directive]

Degree – 1

Q.59 Which term of the series 37+32+27+22+.............. is –103?

(A) 24th **(B)** 30th

(C) 15th **(D)** 29th

Ans: D

a = 37, d = -5

$T_n = a + (n-1)d$

$-103 = 37 + (n-1)(-5)$

$+145 = +5n$

n = 29

Q.60 How many terms are there in the expansion of $\left[(x-5y)^5\right]^3$

(A) 4 **(B)** 6
(C) 16 **(D)** 10

Ans: C

The given expansion is $\{(x-5y)^5\}^3 = (x-5y)^{15}$

No. of terms in the expansion is = 15 + 1 = 16

(ONE more than the power of given expansion)

Q.61 If $\sin\theta = \frac{-3}{5}$ and $\sec\theta = \frac{5}{4}$, find the value of cot θ

(A) $\frac{2}{3}$ **(B)** $\frac{-4}{3}$

(C) $\frac{5}{3}$ **(D)** $\frac{4}{5}$

Ans: B

$\sin\theta = -\frac{3}{5}$, and $\sec\theta = \frac{5}{4} \Rightarrow \cos\theta = \frac{4}{5}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{4/5}{-3/5} = -\frac{4}{3} \Rightarrow \cot\theta = -\frac{4}{3}$

Q.62 Expansion of $5\sin\theta\sin 8\theta$ is equal to

(A) $\frac{5}{2}(\cos 7\theta - \cos 9\theta)$ **(B)** $5(\sin 7\theta + \sin 9\theta)$

(C) $10(\cos 7\theta + \cos 9\theta)$ **(D)** $\frac{2}{5}(\sin 7\theta - \cos 9\theta)$

Ans: A

$5\sin\theta\sin 8\theta = 5/2(2\sin\theta\sin 8\theta)$

$= \frac{5}{2}\{\cos(8\theta-\theta) - \cos(8\theta+\theta)\}$

$= \frac{5}{2}\{\cos 7\theta - \cos 9\theta\}$

Q.63 For what value of k do the points $(-1,4), (-3,8)$ & $(-k+1, 3k)$ lie on a straight line.

(A) 3 **(B)** 4
(C) 0 **(D)** 1

Ans: C

The points say A(-1, 4), B(-3, 8), C(-k + 1, 3k) lies on straight line if area of ΔABC = 0

$$\frac{1}{2}\begin{vmatrix} -1 & 4 & 1 \\ -3 & 8 & 1 \\ -k+1 & 3k & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[-(8-3k)-4(-3+k-1)+1(-9k+8k-8)\right]=0$$

$$\Rightarrow \frac{1}{2}\left[-8+3k+16-4k-k-8\right]=0$$

$$\Rightarrow k=0$$

Q.64 Mid point of the line joining (3, 5) and $(-7,-3)$ is given by

(A) $(-2,1)$ **(B)** (1, 2)

(C) (2, 3) **(D)** (2, 1)

Ans: A

The midpoint of the line joining (3, 5) and (-7, -3) is

$$\left(\frac{3-7}{2},\frac{5-3}{2}\right)=(-2,1) \text{ mid point.}$$

Q.65 $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x}-\sqrt{2}}$ is equal to

(A) $\sqrt{2}$ **(B)** $2\sqrt{2}$

(C) $3\sqrt{2}$ **(D)** $5\sqrt{2}$

Ans: B

$$\lim_{x\to 2}\frac{x-2}{\sqrt{x}-\sqrt{2}}$$

$$\lim_{x\to 2}\frac{(\sqrt{x}+\sqrt{2})(\sqrt{x}-\sqrt{2})}{(\sqrt{x}-\sqrt{2})}$$

$$\lim_{x\to 2}\sqrt{x}+\sqrt{2}=\sqrt{2}+\sqrt{2}=2\sqrt{2}$$

Q.66 If y = x sin x, then $\dfrac{dy}{dx}$ is equal to

(A) cos x + sin x **(B)** cos x + x sin x

(C) x cos x + sin x **(D)** x cos x – sin x

Ans: C

If y = x sin x

Differentiating both side w.r. to x we have

$$\frac{dy}{dx}=x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x\cos x + \sin x$$

Q.67 $\int \tan^2 xdx$ is equal to

(A) $\tan x + c$ **(B)** $\sec^2 x + c$
(C) $x + \tan x + c$ **(D)** $\tan x - x + c$

Ans: D

$$\int \tan^2 xdx = \int (\sec^2 x - 1)dx \Rightarrow \int \sec^2 xdx - \int dx = \tan x - x + c$$

Q.68 The solution of the differential equation $(1+y^2)dx + (1+x^2)dy = 0$ is

(A) $(x + y) = k(1 - xy)$ **(B)** $y - x = kxy$
(C) $x^2 + y = kxy$ **(D)** $y + x = k$

Ans: A

$$(1+a^2)dx + 1(1+x^2)dy = 0$$

Using variable separable method

$$\frac{1}{1+x^2}dx + \frac{1}{1+y^2}dy = 0$$

$$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1} k$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} k$$

$$\frac{x+y}{1-xy} = k \Rightarrow x + y = k(1-xy)$$

Q.69 The square root of $12 - 5i$ is

(A) $\pm\frac{1}{2}(5-i)$. **(B)** $\pm\frac{1}{2}(5+i)$.
(C) $\pm\frac{1}{\sqrt{2}}(5-i)$. **(D)** $\pm\frac{1}{\sqrt{2}}(5+i)$.

Ans: C

Let $\sqrt{12-5i} = \pm(x+iy)$

$$12\text{-}5i = (x+iy)^2 = x^2 - y^2 + 2ixy \Rightarrow x^2 - y^2 = 12, \quad 2xy = -5 \Rightarrow y = \frac{-5}{2x}$$

$$\Rightarrow x^2 - \frac{25}{4x^2} = 12 \Rightarrow 4x^4 - 48x^2 - 25 = 0 \Rightarrow x^2 = \frac{100}{8}, \ \frac{-4}{8}$$

$$\Rightarrow x^2 = \frac{25}{2} \quad \left[\because x^2 \neq \frac{-4}{8}\right] \Rightarrow x = \frac{\pm 5}{\sqrt{2}}, \ y = \frac{\mp 5\sqrt{2}}{2\times 5} = \frac{\mp\sqrt{2}}{2} = \frac{\mp 1}{\sqrt{2}}$$

$\therefore \sqrt{12-5i} = \pm\left(\frac{5}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \pm\left(\frac{5-i}{\sqrt{2}}\right)$

Q.70 If α, β be the roots of $ax^2 + bx + c = 0, a \neq 0,$ then the quadratic equation whose roots are α^2, β^2 is

(A) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$. **(B)** $a^2x^2 + (b^2 - 2ac)x + c^2 = 0$.
(C) $c^2x^2 - (b^2 - 2ac)x + a^2 = 0$. **(D)** $c^2x^2 + (b^2 - 2ac)x + a^2 = 0$.

Ans: A

We know that $\alpha + \beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$. Since $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$,

$\alpha^2\beta^2 = \frac{c^2}{a^2}$ equation is $x^2 - \left(\frac{b^2 - 2ac}{a^2}\right)x + \frac{c^2}{a^2} = 0$

or $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$

Q.71 The 12th term in the binomial expansion of $\left(x - \frac{1}{x}\right)^{15}$ is

(A) $15_{C12}\ x^{-9}$. **(B)** $-15_{C11}\ x^{-7}$.
(C) $15_{C12}\ x^{9}$. **(D)** $-15_{C11}\ x^{7}$.

Ans: B
12th term in the expansion of
$\left(x - \frac{1}{x}\right)^{15}$ is ${}^{15}c_{11}\ x^4\left(-\frac{1}{x}\right)^{11} = -{}^{15}c_{11}\ x^{-7}$

Q.72 The area of the triangle formed by the coordinate axes and the line 2x + 3y=6 is
(A) 3 sq. units. **(B)** 6 sq. units.
(C) 9 sq. units. **(D)** 12 sq. units.

Ans: A

Area of triangle is $\frac{1}{2}$ base $\times$ height $= \frac{1}{2} \times 3 \times 2 = 3$ sq. unit

Q.73 The eccentricity of the ellipse if its latus rectum is equal to one half of its minor axis is

(A) $\sqrt{3}$. **(B)** $\frac{\sqrt{3}}{2}$.

(C) ½. **(D)** $\frac{1}{\sqrt{3}}$.

Ans: B

Eccentricity $= \frac{\sqrt{3}}{2}$

Q.74 In a triangle ABC, sin A – cos B = cos C, then angle B is

(A) $\frac{\pi}{2}$. **(B)** $\frac{\pi}{3}$.

(C) $\frac{\pi}{4}$. **(D)** $\frac{\pi}{6}$.

Ans: A

Given sin A-cos B=cos C $\Rightarrow$ Sin A= cos B+ cos C $=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$$2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) \quad =2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\cos\frac{A}{2}=\cos\left(\frac{B-C}{2}\right) \quad \Rightarrow\frac{A}{2}=\frac{B-C}{2} \quad \Rightarrow A=B-C \quad \Rightarrow B=A+C \quad =\pi-B$$

$$2B=\pi\Rightarrow B=\frac{\pi}{2}$$

Q.75 $\int_1^e \log x \ dx$ is equal to

(A) e – 1. **(B)** e + 1.
(C) 0. **(D)** 1.

Ans: D

$$\int_1^e 1\cdot\log x dx=\left[\log x\cdot x\Big|_1^e-\int_1^e\frac{1}{x}x dx\right] \quad =e\log e-\log 1-[x]_1^e \quad \text{=e-e+1} \quad =1$$

Q.76 If $x^y=e^{x-y}$, then $\frac{dy}{dx}$ is equal to

(A) $(1+\log x)^{-1}$. **(B)** $(1+\log x)^{-2}$

(C) $\log x\,(1+\log x)^{-2}$. **(D)** $\log x\,(1+\log x)^{-1}$.

Ans: C

Given $x^y=e^{x-y}$ $\Rightarrow y\log x=x-y$

$$\Rightarrow y[\log x+1]=x \quad \Rightarrow y=\frac{x}{1+\log x}$$

$$\therefore\frac{dy}{dx}=\frac{(1+\log x)\cdot 1-x\left[\frac{1}{x}\right]}{(1+\log x)^2} \quad =\frac{1+\log x-1}{(1+\log x)^2} \quad \text{or} \quad \frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

Q.77 The point (α,β) is equidistant from points (7,6) and (-3, 4) if

(A) $\alpha=3 \quad \beta=0$. **(B)** $\alpha=0 \quad \beta=3$.
(C) $\alpha=3=\beta$. **(D)** $\alpha=0=\beta$.

Ans: A

$(\alpha-7)^2+(\beta-6)^2=(\alpha+3)^2+(\beta-4)^2$

$\alpha^2-14\alpha+49+\beta^2-12\beta+36$

$=\alpha^2+6\alpha+9+\beta^2-8\beta+16$

$\Rightarrow -20\alpha-4\beta+60=0 \Rightarrow 5\alpha+\beta-15=0 \quad \Rightarrow \alpha=3, \beta=0$

Q.78 The value of $\sin(45°+\theta)-\cos(45°-\theta)$ is

(A) $2\cos\theta$ **(B)** $2\sin\theta$

(C) 1 **(D)** 0

Ans: D

$\sin(45^0+\theta)-\cos(45^0-\theta)$

$= \sin 45^0 \cos\theta+\cos 45^0 \sin\theta$ - $\cos 45^0 \cos\theta-\sin 45^0 \sin\theta$

$=\frac{\sin\theta}{\sqrt{2}}+\frac{\cos\theta}{\sqrt{2}}-\frac{\cos\theta}{\sqrt{2}}-\frac{\sin\theta}{\sqrt{2}}=0$

Q.79 The equation of a line through point (2, -3) and parallel to y-axis is

(A) y = -3 . **(B)** y = 2.

(C) x = 2. **(D)** x = -3.

Ans: C

The equation of line parallel to y-axis and at a distance 2 is $x=2$

Q.80 The length of tangent from point (5,1) to the circle $x^2+y^2+6x-4y-3=0$ is

(A) 81. **(B)** 29.

(C) 7. **(D)** 21.

Ans: C

Here S = $x^2+y^2+6x-4y-3=0$

$\Rightarrow S_1=25+1+30-4-3$

$=49$

length of tangent $=\sqrt{49}=7$

Q.81 The differential coefficient of log tan x is

(A) 2 sec2 x. **(B)** 2 cosec 2 x.

(C) 2 $\sec^3$ x. **(D)** 2 cosec^3 x.

Ans: B

Let y = log tan x

$$\frac{dy}{dx}=\frac{1}{\tan x}\cdot\sec^2 x=\frac{1}{\cos x\sin x}=2\cos ec2x$$

Q.82 The expression $(3+w+3w^2)^4$ where w is a cube root of unity, equals

(A) 16. **(B)** 16 w.

(C) 16 w^2 . **(D)** 0.

Ans: B

$(3 - w + 3w^2)^4$

$= [3(1+w^2)+w]^4 = (-3w+w)^4 = (-2w)^4 = 16w^4$

$= 16w$

Q.83 The complex number z = x +iy which satisfies the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lie on

(A) The x-axis.
(B) The straight line y =5.
(C) A circle passing through the origin.
(D) The y-axis.

Ans: A

$\left|\frac{z-5i}{z+5i}\right| = 1$

$|z-5i| = |z+5i|$

$|x+iy-5i| = |x+iy+5i|$

$|x+i(y-5)| = |x+i(y+5)| \Rightarrow x^2+(y-5)^2 = x^2+(y+5)^2$

$\Rightarrow$ -10 y + 25 = 10 y + 25

y =0 $\Rightarrow x - axis$

Q.84 If $a^x = b,\ b^y = c,\ c^z = a$, then the value of xyz is

(A) 0. **(B)** 1.
(C) 2. **(D)** 3.

Ans: B

Given $a^x = b$

$b^y = c$

$c^z = a$

$a = c^z = (b^y)^z = b^{yz} = (a^x)^{yz} = a^{xyz} \Rightarrow xyz = 1$

Q.85 The equation whose roots are the reciprocals of the roots of the equation $ax^2+bx+c=0$ is

(A) $\frac{x^2}{a}+\frac{x}{b}+\frac{1}{c}=0$. **(B)** $bx^2+cx+a=0$.

(C) $ax+b+cx^2=0$. **(D)** $a+bx+cx^2=0$.

Ans: D

We have $ax^2+bx+c=0$ --------(1)

Let α, β are roots of (1), then $\alpha+\beta = -\frac{-b}{a},\ \alpha\beta = \frac{c}{a}$

Again $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-b}{a}\times\frac{a}{c} = \frac{-b}{c}$ and $\frac{1}{\alpha}.\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{a}{c}$

Equation is $x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0 \;\Rightarrow x^2 + \frac{b}{c}x + \frac{a}{c} = 0$

$cx^2 + bx + a = 0$

Q.86 The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is

(A) 2 **(B)** 4
(C) 3 **(D)** 5

Ans: B

We have

$\frac{1+i}{1-i} = \frac{(1+i)^2}{1+1} = i \qquad \therefore \left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow i^n = 1$

$\Rightarrow$ n is a multiple of 4

$\Rightarrow$ the smallest positive value of n is 4

Q.87 If $\cos^2\theta + 3\sin\theta = 3$, then θ is equal to

(A) 90°. **(B)** 60°.
(C) 45°. **(D)** 30°.

Ans: A

Given $\cos^2\theta + 3\sin\theta = 3$

$1 - \sin^2\theta + 3\sin\theta - 3 = 0$

$\sin^2\theta - 3\sin\theta + 2 = 0$

$\sin\theta = \frac{3 \pm \sqrt{9-8}}{2} = 2,1,\ \sin\theta \neq 2$

$\theta = \sin^{-1}1 = 90^0$

Q.88 $\int_o^a f(x)dx$ is equal to

(A) $\int_o^a f(a+x)dx$. **(B)** $\int_o^a f(a-x)dx$.
(C) $\int_o^a f(ax)dx$. **(D)** $\int_o^a f\left(\frac{x}{a}\right)dx$.

Ans: B

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Q.89 Integrating factor of the differential equation $(1+x)\frac{dy}{dx} - y = (1+x)^2$ is

(A) 1 + x. **(B)** e^{1+x}.

(C) $\frac{1}{1+x}$. **(D)** e^{-x}.

Ans: C

$$(1+x)\frac{dy}{dx} - y = (1+x)^2 \quad \Rightarrow \quad \frac{dy}{dx} - \frac{1}{1+x}y = 1+x$$

$$\text{I.F.} = e^{-\int \frac{1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{1+x}$$

Q.90 The distance between two parallel lines 3x + 4y = 5 and 6x + 8y = 35 is

(A) 1.0. **(B)** 1.5.
(C) 2.0. **(D)** 2.5.

Ans: D

Putting y=0 in $3x+4y=5$ we get $x = 5/3$

Thus $(5/3, 0)$ lie on $3x+4y=5$

The length of perpendicular from $(5/3, 0)$ to 6x+8y =35 is

d= $\frac{|6(5/3)+8(0)-35|}{\sqrt{6^2+8^2}} = \frac{25}{10} = 2.5$ Hence, the distance between the given lines is 2.5

Q.91 The angle between the vectors $\vec{A} = 2i + j - 3k$ and $\vec{B} = 3i - 2j - k$ is

(A) 30°. **(B)** 45°.

(C) 60°. **(D)** 90°.

Ans: C

We know that :

$$\bar{A} \cdot \bar{B} = |\bar{A}| . |\bar{B}| \cos\theta \Rightarrow \cos\theta = \frac{\overline{A}.\overline{B}}{|\bar{A}||\bar{B}|}$$

$$\frac{(2i+j-3k).(3i-2j-k)}{|2i+j-3k||3i-2j-k|}$$

$$= \frac{6-2+3}{\sqrt{4+1+9}\sqrt{9+4+1}} = \frac{7}{\sqrt{14}\sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\frac{1}{2} = 60^0$$

Q.92 The value of $\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$ is

(A) 2. **(B)** 4.
(C) 6. **(D)** zero.

Ans: A

$$\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{x \to 0} \frac{1-(1-2\sin^2 x)}{x^2} \qquad = \lim_{x \to 0} \frac{2\sin^2 x}{x^2}$$

$$= 2\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \qquad = 2.1 \quad = 2$$

Q.93 $1^2 + 2^2 + 3^2 + ... + n^2$ is equal to

(A) $\frac{n(n+1)}{2}$. **(B)** $\frac{n(2n+1)}{2}$.
(C) $\frac{n(n+1)(2n+1)}{6}$. **(D)** $\frac{(n+1)(2n+1)}{6}$.

Ans: C

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)\,(2n+1)}{6}$$

Q.94 If $\log_9 x = 1.5$ then x is equal to

(A) 3 **(B)** 27
(C) 9 **(D)** 15

Ans: B

Given $\log_9 x = 1.5 \;=\; x = 9^{1.5} \;= (3^2)^{1.5} = 3^{3.0} \;= 27$

Q.95 The value of $\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$ is equal to

(A) 1. **(B)** 0.
(C) -1. **(D)** ½

Ans: A

$$\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ = \sin(75^\circ + 15^\circ) = \sin(90^\circ)$$

Q.96 If α, β are the roots of $ax^2 + bx + c = 0$ then $\alpha^2 + \beta^2$ is

(A) $-b/c$. **(B)** $\frac{b^2 - 2ac}{a^2}$.

(C) $b^2 - 2ac$. **(D)** $\frac{b^2 - ac}{a^2}$.

Ans: B

Since α, β are roots of $ax^2 + bx + c = 0$

$$\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$$

Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \quad = \frac{b^2}{a^2} - \frac{2c}{a} \;= \frac{b^2 - 2ac}{a^2}$

Q.97 $\lim_{x \to 1} (x)^{\frac{1}{(x-1)}}$ is equal to

(A) 1. **(B)** 0.

(C) e. **(D)** $\frac{1}{e}$.

Ans: C

$$y = \lim_{x \to 1} x^{\frac{1}{x-1}} \Rightarrow \log y = \lim_{x \to 1} \frac{1}{x-1} \log x = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

$$\log y = 1 \Rightarrow y = e$$

Q.98 $\int \log x \, dx$ is equal to

(A) $x \log x - x + c$. **(B)** $x \log x$.

(C) $\log x$. **(D)** $\frac{1}{x} \log x$.

Ans: A

$$\int \log x \, dx = \int 1.\log x dx \quad = \log x.x - \int \frac{1}{x}.x dx + c \;= x.\log x - x + c$$

Q.99 The maximum value of $y = 2\cos 2x - \cos 4x,\ 0 \le x \le \pi/2$ is

(A) -1. **(B)** $1/2$.

(C) $3/2$. **(D)** 1.

Ans: C

$y = 2\cos 2x - \cos 4x, o \le x \le \frac{\pi}{2}, \quad \frac{dy}{dx} = -4\sin 2x + 4\sin 4x$

For maxima and minima $\frac{dy}{dx} = 0 \Rightarrow \sin 2x - \sin 4x = 0 \Rightarrow \sin 2x - 2\sin 2x\cos 2x = 0$

$\Rightarrow \sin 2x = 0$ or $(1 - 2\cos 2x) = 0. \sin 2x = 0 \Rightarrow x = 0$

$1 - 2\cos 2x = 0 \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

$\frac{d^2y}{dx^2} = -8\cos 2x + 16\cos 4x. \left(\frac{d^2y}{dx^2}\right)_{x=0} = -8 + 16 = 8 > 0$

$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{6}} = -4 - 8 = -12 < 0$ Thus y is a maximum at $x = \frac{\pi}{6}$

, Maximum value is $y = 2\cos\frac{\pi}{3} - \cos\frac{2\pi}{3} = \frac{3}{2}$ Ans : C

Q.100 The equation of the line which is perpendicular to the line 3x – 4y +7=0 and passes through the point (-3, 2) is

(A) 4x + 3y + 5 = 0. **(B)** 4x + 3y –3 = 0.
(C) 4x + 3y + 6 = 0. **(D)** 3x – 4y + 6 = 0.

Ans: C

The equation of line perpendicular to

$3x - 4y + 7 = 0$ is $-4x - 3y + \lambda = 0$ ________(1)

This passes through (-3,2)

$\therefore -4(-3) - 3(2) + \lambda = 0 = 12 - 6 + \lambda = 0 = 6 + \lambda = 0 = \lambda = -6$

From (i), required equation is

$-4x - 3y - 6 = 0 = 4x + 3y + 6 = 0$

PART – II

NUMERICALS

Q.1 If α, β are the roots of the equation $ax^2 + bx + c = 0$. Find the equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$. **(7)**

Ans:

$$\alpha + \beta = -\frac{b}{a} \qquad \therefore \quad (\alpha + \beta)^2 = \frac{b^2}{a^2}$$

$$\alpha.\beta = \frac{c}{a}$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$= \frac{b^2}{a^2} - 4\frac{c}{a}$$

$$= \frac{b^2 - 4ac}{a^2}$$

$$\therefore \quad (\alpha + \beta)^2 + (\alpha - \beta)^2 = \frac{b^2 - 4ac}{a^2} + \frac{b^2}{a^2}$$

$$= \frac{2b^2 - 4ac}{a^2}$$

$$\therefore \quad (\alpha + \beta)^2.(\alpha - \beta)^2 = \frac{b^2 - 4ac}{a^2}.\frac{b^2}{a^2}$$

Therefore required equation is

$$x^2 - \frac{2b^2 - 4ac}{a^2}x + \frac{(b^2 - 4ac)b^2}{a^4} = 0$$

$$\approx a^4x^2 - a^2(2b^2 - 4ac)x + (b^2 - 4ac)b^2 = 0$$

Q.2 If the roots of the equation $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$ are equal, show that $\frac{1}{p} + \frac{1}{r} = \frac{2}{q}$. **(7)**

Ans:

$$q^2(r - p)^2 = 4pr(q - r)(p - q)$$

$$q^2(r^2 + p^2 - 2pr) = 4pr(pq - q^2 - rp + rq)$$

$$\sim q^2(r^2 + p^2 + 2pr) = 4pr(pq - q^2 - rp + rq)$$

$$\sim \frac{1}{p^2} + \frac{1}{r^2} + \frac{2}{pr} = \frac{4}{rq} + \frac{4}{pq} - \frac{4}{q^2}$$

$$\sim \frac{1}{p^2} + \frac{1}{r^2} + \frac{4}{q^2} - \frac{4}{rq} - \frac{4}{pq} + \frac{2}{pr} = 0$$

$$\Rightarrow \left(\frac{1}{p}+\frac{1}{r}-\frac{2}{q}\right)^2 = 0$$

$$\Rightarrow \frac{1}{p}+\frac{1}{r}=\frac{2}{q}$$

Q.3 In a $\Delta\,ABC$ show that $c^2 = (a-b)^2 \cos^2\frac{C}{2} + (a+b)^2 \sin^2\frac{C}{2}$. **(7)**

Ans:

$$\text{R.H.S.} = K^2(\sin A-\sin B)^2\cos^2\frac{c}{2}+K^2(\sin A+\sin B)^2\sin^2\frac{c}{2}$$

$$= K\left[4\cos^2\frac{A+B}{2}.\sin^2\frac{A-B}{2}.\cos^2\frac{c}{2}+4\sin^2\frac{A+B}{2}.\cos^2\frac{A-B}{2}.\sin^2\frac{c}{2}\right]$$

$$= 4K^2\left[\sin^2\frac{c}{2}.\cos^2\frac{c}{2}.\sin^2\frac{A-B}{2}+\cos^2\frac{c}{2}.\sin^2\frac{c}{2}.\cos^2\frac{A-B}{2}\right]$$

$$= K^2\sin^2 c = c^2$$

Q.4 If $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$ then show that $x=\frac{a+b}{1-ab}$. **(7)**

Ans:

$$\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$$

$$\Rightarrow \quad 2\tan^{-1}a+2\tan^{-1}b=2\tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}a+\tan^{-1}b=\tan^{-1}x$$

$$\Rightarrow \quad x=\frac{a+b}{1-ab}$$

Q.5 Evaluate $\lim\limits_{x\to 0}\frac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x^2}$. **(7)**

Ans:

$$\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x^2}$$

$$= \lim_{x\to 0}\frac{a^2+x^2-a^2+x^2}{x^2\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)}$$

$$= \lim_{x\to 0}\frac{2}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$$

$$= \frac{2}{a+a}=\frac{1}{a}$$

Q.6 Differentiate $f(x)=\mathrm{Cos}^2x$ by the first principle. **(7)**

Ans:

$f(x) = \cos^2 x$

$f(x + \delta x) = \cos^2(x + \delta x)$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\cos^2(x+\delta x) - \cos^2 x}{\delta x}$$

$$\frac{\delta x}{\delta y} = \lim_{\delta x \to 0} \frac{[\cos(x+\delta x) - \cos x][\cos(x+\delta x) + \cos x]}{\delta x}$$

$$\frac{\delta x}{\delta y} = \lim_{\delta x \to 0} \frac{2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{-\delta x}{2}\right)(\cos(x+\delta x) + \cos x)}{\delta x}$$

$$= -\lim_{\delta x \to 0} 2\sin\left(x + \frac{\delta x}{2}\right)\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}.2}\right)(\cos(x+\delta x) + \cos x)$$

$$= -2\sin x.\frac{1}{2}.2\cos x$$

$= -\sin 2x.$

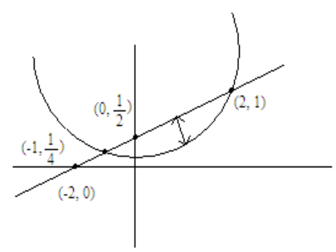
Q.7 Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. **(7)**

Ans:

Area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

$$= \iint dy\,dx$$

The above curve intersects at the points $\left(-1, \frac{1}{4}\right)$ and (2, 1).



$$\int_{-1}^{1}\int_{\frac{x^2}{4}}^{\frac{x+2}{4}} dy\,dx = \int_{-1}^{1}[y]_{\frac{x^2}{4}}^{\frac{x+2}{4}}\,dx$$

$$= \int_{-1}^{1}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \frac{1}{4}\left(\frac{x^2}{2} + 2x\right)_{-1}^{1} - \left(\frac{x^3}{12}\right)_{-1}^{1}$$

$$= \frac{1}{4}(4) - \left(\frac{1}{12} + \frac{1}{12}\right) = 1 - \frac{1}{6} = \frac{5}{6} \text{ Units.}$$

Q.8 Find the equation of tangent to $16x^2 + 9y^2 = 144$ at (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$. **(7)**

Ans:

Equation of the given ellipse is $16x^2 + 9y^2 = 144$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1$$

Tangent at the point $(2, y_1)$ such as $y_1 > 0$.

Equation of tangent at point (x_1, y_1) is $\Rightarrow \frac{xx_1}{9}+\frac{yy_1}{16}=1$

Satisfies the point (2, y_1) $\Rightarrow \frac{4}{9}+\frac{y^2}{16}=1 \Rightarrow \frac{y^2}{16}=\frac{5}{9}$

$\Rightarrow y_1=\pm\frac{4}{3}\sqrt{5}$, $y_1 > 0$

The equation of tangent at $\left(2,\frac{4}{3}\sqrt{5}\right)$ is

$$\frac{2x}{9}+\frac{\frac{4}{3}\sqrt{5}}{16}y=1$$

$$\frac{2x}{9}+\frac{\sqrt{5}}{12}y=1$$

Q.9 Find the equation of a line passing through $(-2,-4)$ and perpendicular to the line 3x –y +5 = 0. **(7)**

Ans:

Let the equation of line is y = wx + c ……………….(1)
because it is perpendicular to 3x – y + 5 = 0

$\therefore\ 3w=-1 \quad \therefore\ w=\frac{-1}{3}$

Therefore (1) becomes

$$y=\frac{-1}{3}x+c$$

~ $3y+x=3c$

It is passing through the point (-2, -4) therefore

-12 -2 = 3c

~ -14 = 3c

$\Rightarrow c=\frac{-14}{3}$

$\therefore$ required equation is

$$3y+x=\frac{-14}{3}.3$$

x + 3y + 14 = 0

Q.10 Find the equation of the circle whose centre lies on the line x – 4y = 1 and which passes through the points (3, 7) and (5,5). **(7)**

Ans:

Let the equation of the circle is

$(x-h)^2+(y-k)^2=r^2$

The centre lies on the line x – 4y = 1,

$\therefore$ h – 4k = 1 …….(1)

Again the circle passes through (3, 7) and (5, 5)

$(3-h)^2+(7-k)^2=r^2=(5-h)^2+(5-k)^2$

$\Rightarrow h^2-6h+9+k^2-14k+49=25-10h+h^2+25-10k+k^2=r^2$

$\Rightarrow 4h-4k=-8$

h – k = -2 ……..(2)

Subtracting (1) from equation (2)

3k = -3 $\Rightarrow$ k = -1

and h = -3

putting the value of h and k , we have

$36+64=r^2 \sim r=10$

$\therefore$ required equation is

$(x+3)^2+(y+1)^2=(10)^2$

$\sim x^2+y^2+6x+2y+10=100$

$\Rightarrow x^2+y^2+6x+2y-90=0$

Q.11 Find the term independent of x in the expansion of $\left(2x-\frac{1}{x}\right)^{10}$. **(7)**

Ans:

$\left(2x-\frac{1}{x}\right)^{10}$

Middle term is independent from x i.e.

$10C_5(2x)^5\frac{1}{x^5}$

$=\frac{\lfloor 10}{\lfloor 5\lfloor 5}2^5$

$=\frac{6\times7\times8\times9}{1.2.3.4.5}2^5$

= 8064.

Q.12 Evaluate $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$. **(7)**

Ans:

$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

$\therefore 2I=\int_0^{\pi/2}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

$=\frac{\pi}{2}$

$I=\frac{\pi}{4}$

Q.13 Using induction, prove that $2^n>n$ for all n. **(7)**

Ans:

By using mathematical induction method

$2^1 > 1$

This is true n = 1

Let $2^r > r$

Now $2^{r+1} = 2.2^r > 2r > r+1$ if $r > 1$

Therefore on the statement is true for r = n

Hence it is true for all n.

Q.14 Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$. **(7)**

Ans:

$$\frac{dy}{dx} = e^x.e^{-y} + x^2 e^{-y}$$

$$e^y dy = (e^x + x^2)dx$$

$$e^y = e^x + \frac{x^3}{3} + c$$

$$\Rightarrow Y = \log\left[e^x + \frac{x^3}{3} + c\right]$$

Q.15 Evaluate $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$. **(7)**

Ans:

Let $\tan^{-1} x^3 = t$

$$\Rightarrow \frac{1}{1+x^6} 3x^2 dx = dt$$

$$\therefore \quad I = \frac{1}{3}\int t dt = \frac{t^2}{6} + c$$

$$= \frac{(\tan^{-1} x^3)}{6} + c$$

Q.16 Evaluate $\int \frac{x - \sin x}{1 - \cos x} dx$. **(7)**

Ans:

$$\int \frac{x - \sin x}{1 - \cos x} dx$$

$$= \int \frac{x}{1-\cos x} dx - \int \frac{\sin x}{1-\cos x} dx$$

$$= \int \frac{x}{2\sin^2 \frac{x}{2}} dx - \log(1 - \cos x)$$

$$= \frac{1}{2}\int x \cos ec^2 \frac{x}{2} dx - \log(1-\cos x) + c$$

$$= -\frac{1}{2}\frac{x\cot\frac{x}{2}}{\frac{1}{2}} + \int \cot\frac{x}{2} - \log(1-\cos x) + c$$

$$= -x\cot\frac{x}{2} + 2\log\sin\frac{x}{2} - \log 2\sin^2\frac{x}{2} + c$$

$$= -x\cot\frac{x}{2} - \log 2 + c$$

Q.17 Solve $\left(x^2 - y^2\right)dx + 2\,xy\,dy = 0$, given y =1 when x = 1. **(7)**

Ans:

$$(x^2 - y^2)dx + 2xydy = 0$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = -\frac{(v^2 + 1)}{2v}$$

$$\frac{dx}{x} = -\frac{2v}{v^2 + 1}dv$$

$$\log x = -\log(v^2 + 1) + \log c$$

$$= -\log\left(\frac{x^2}{y^2} + 1\right) + \log c$$

When y = 1, x = 1

$$\therefore \quad 0 = -\log\left(\frac{1}{1} + 1\right) + \log c$$

c = 2

$$\therefore \quad \log x + \log\frac{(y^2 + x^2)}{x^2} = \log 2$$

$$\log\frac{(y^2 + x^2)}{x} = \log 2$$

$$y^2 + x^2 = 2x$$

$$\Rightarrow x^2 + y^2 - 2x = 0$$

Q.18 Find the differential equation of which $y = Ae^{x} + Be^{3x} + Ce^{5x}$ is a solution. **(7)**

Ans:

$$Y = Ae^{x} + Be^{3x} + Ce^{5x}$$

$$\frac{dy}{dx} = y_1 = Ae^x + 3Be^{3x} + 5Ce^{5x}$$
$$\frac{d^2y}{dx^2} = y_2 = Ae^x + 9Be^{3x} + 25Ce^{5x}$$
$$\frac{d^3y}{dx^3} = y_3 = Ae^x + 27Be^{3x} + 125Ce^{5x}$$
$$\Rightarrow \quad (Ae^x + 27Be^{3x} + 125Ce^{5x}) - 9(Ae^x + 9Be^{3x} + 25Ce^{5x})$$
$$+ 23(Ae^x + 3Be^{3x} + 5Ce^{5x}) - 15(Ae^x + Be^{3x} + Ce^{5x}) = 0$$
$$\Rightarrow \quad \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$$

Q.19 Find the term independent of x in the expansion of $(x - \frac{1}{x})^{12}$. **(8)**

Ans:

$$T_{n+1} = {}^{12}C_n x^{12-n}\left(-\frac{1}{x}\right)^n = {}^{12}C_n (-1)^n x^{12-2n}$$

If n^{th} term is independent of x

12 – 2n = 0 i.e. n = 6

$\therefore T_{6+1} = T_7$ is independent of x and

$T_7 = {}^{12}C_6 (-1)^6 = {}^{12}C_6 = 924$

Q.20 If the p^{th}, q^{th} and r^{th} terms of an A.P. are x, y, z respectively, show that x (q – r) + y (r – p) + z (p – q) = 0. **(8)**

Ans:

If a, a + d, a + 2d, ………… be A.P.,

$T_p = x = a + (p - 1)\,d$

$T_q = y = a + (q - 1)\,d$

$T_r = z = a + (r - 1)\,d$

$\therefore x(q-r) + y(r-p) + z(p-q)$

$= a(q - r + r - p + p - q) + d[(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)]$

$= d[qp - q - rp + r + qr - qp - r + p + rp - rq - p + q] = 0$

Q.21 If A + B + C = π, show that

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$ **(8)**

Ans:

A+B+C = π or $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$\therefore \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}$

$$or\ \frac{\tan A/2+\tan B/2}{1-\tan A/2.\tan B/2}=\frac{1}{\tan C/2}$$

By cross multiplying

$$\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$or\ \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

Dividing through out by $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$, we get

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

Q.22 In any triangle ABC, show that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$ **(8)**

Ans:

In any triangle ABC,

$A + B + C = \pi$

and $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=K$ (say)

$$\therefore\ \frac{b-c}{b+c}=\frac{K\sin B-K\sin C}{K\sin B+K\sin C}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\frac{\tan\frac{B-C}{2}}{\tan\left(\frac{B}{2}+\frac{C}{2}\right)}=\frac{\tan\frac{B-C}{2}}{\tan\left(\frac{\pi}{2}-\frac{A}{2}\right)}$$

$$=\frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}$$

Hence $\frac{b-c}{b+c}\cot\frac{A}{2}=\tan\frac{B-C}{2}$

Q.23 Solve the equation $x\frac{dy}{dx}-3y=x^2$. **(8)**

Ans:

$$x\frac{dy}{dx}-3y=x^2 \quad \Rightarrow\frac{dy}{dx}-\frac{3}{x}y=x$$

I.F. = $e^{-\int \frac{3}{x} dx} = e^{-3\log x} = x^{-3}$

Solution is

$$yx^{-3} = \int x^{-3}\, x\, dx + c = \int x^{-2}\, dx + c$$

$$yx^{-3} = \frac{x^{-1}}{-1} + c$$

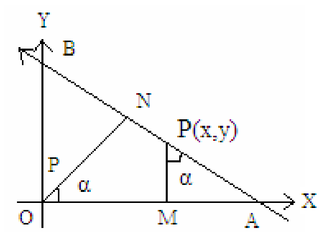
$$y = -x^2 + cx^3$$

Q.24 Find the equation of a straight line when p is the length of perpendicular on it from the origin and the inclination of this perpendicular to the x – axis is α. **(8)**

Ans:

Let ON = p be length of perpendicular from origin on st line AB and let ON make angle α with x-axis.

∴ p = ON = OA cos α
= (OM + MA) cos α
= (x + MP tan α) cos α
= x cos α + y sin α
Hence required equation is
x cos α + y sin α = p



Q.25 Find the equation of the straight line which passes through the intersection of the straight lines 2x – 3y + 4 = 0 and 3x + 4y + 5 = 0 and is perpendicular to the straight line 6x – 7y + 8 = 0. **(8)**

Ans:

Any line through the intersection of two given lines in
2x – 3y + 4 + k(3x + 4y + 5) = 0
It is perpendicular to the line 6x – 7y + 8 = 0

$$\therefore \left(-\frac{2+3k}{-3+4k}\right)\cdot\left(-\frac{6}{-7}\right) = -1$$

$$\therefore \quad 12 + 18k = -21 + 28k \; or \; 10k = 33, \; k = \frac{33}{10}$$

Hence required straight line in $(2x - 3y + 4) + \frac{33}{10}(3x + 4y + 5) = 0$

Or $119x + 102y + 205 = 0$

Q.26 Show that $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle. Find its centre and radius. **(6)**

Ans:

Given equation can be written as $(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

Or $[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2$. Comparing with $(x-h)^2+(y-k)^2=a^2$ which is a circle of centre (h, k) and radius a, we observe that given equation represents a circle with centre = (-g, -f), Radius = $\sqrt{g^2+f^2-c}$

Q.27 Find the vertex, focus, latus rectum and directrix of the parabola $x^2 = 4x - y$. **(10)**

Ans:

$x^2=4x-y$ or $x^2-4x=-y$

Or $(x-2)^2=-y+4=-(y-4)$

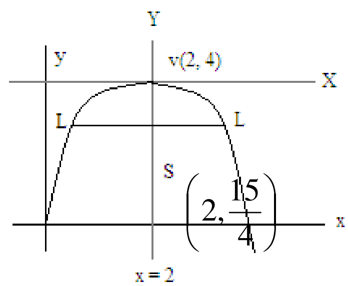
Put x – 2 = X, y – 4 = Y

$\therefore \quad X^2=-Y$ represents a parabola of the shape as shown below.

With vertex X = 0, Y = 0 i.e. x = 2, y = 4, axis x = 2,

LR = 4a = 1.

$\therefore a=\frac{1}{4},\ focus=\left(2,4-\frac{1}{4}\right)=\left(2,\frac{15}{4}\right)$ and Directrix is $y=4+\frac{1}{4}=\frac{17}{4}$



Q.28 Evaluate $\lim\limits_{x\to 0}\frac{a^x-1}{x}$, by using the fact that $\lim\limits_{t\to 0}(1+t)^{1/t}=e$. **(8)**

Ans:

Put $a^x - 1 = t$ or $a^x = t + 1$ or $x = \log_a(1 + t)$ $\therefore as\ x\to 0, t\to 0$

$$\therefore \lim_{x\to 0}\frac{a^x-1}{x}=\lim_{t\to 0}\frac{t}{\frac{\log_e(1+t)}{\log_e a}}=\lim_{t\to 0}\frac{t.\log_e a}{\log_e(1+t)}=\lim_{t\to 0}\frac{\log_e a}{\frac{1}{t}.\log_e(1+t)}$$

$$=\lim_{t\to 0}\frac{\log_e a}{\log_e(1+t)^{1/t}}=\log_e a \text{ because } \lim_{t\to 0}(1+t)^{1/t}=e$$

Q.29 Differentiate $\tan^{-1}\sqrt{\frac{1-\text{Cos}x}{1+\text{Cos}x}}$ with respect to x. **(8)**

Ans:

$$y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}=\tan^{-1}\sqrt{\frac{1-(1-2\sin^2 x/2)}{1+\left(2\cos^2\frac{x}{2}-1\right)}}=\tan^{-1}\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}$$

$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$

Hence $\frac{dy}{dx} = \frac{1}{2}$

Q.30 Find the points at which the function $y = 3\,\text{Sin}^2x + 4\,\text{Cos}^2x$ has maximum and minimum values in the interval $\left[0, \frac{\pi}{2}\right]$ **(8)**

Ans:

$y = 3\sin^2 x + 4\cos^2 x$

$\therefore \frac{dy}{dx} = 6\sin x\cos x + 4.2\cos x(-\sin x) = -2\sin x\cos x = -\sin 2x$

For Max or Min $\frac{dy}{dx} = 0, \therefore \sin 2x = 0 \text{ or } 2x = 0, \pi \text{ or } x = 0, \frac{\pi}{2}$

$\therefore$ points of maximum & minimum are $x = 0, \frac{\pi}{2}$

$$\frac{d^2y}{dx^2} = -2\cos 2x = \begin{cases} -2 \text{ at } x = 0 \\ +2 \text{ at } x = \frac{\pi}{2} \end{cases}$$

Hence x = 0 is a point of Maxima and Max. value is 4

$x = \frac{\pi}{2}$ is a point of Minima and Minimum value is 3.

Q.31 Evaluate $\int \frac{\text{dx}}{\text{a}\cos\text{x} + \text{b}\sin\text{x}}$, where a, b are not both zero. **(8)**

Ans:

$\int \frac{dx}{a\cos x + b\sin x}$ put $a = r\sin\varphi, b = r\cos\varphi$

$\therefore r = \sqrt{a^2 + b^2}, \varphi = \tan^{-1}\frac{a}{b}$

$= \int \frac{dx}{r\sin(\varphi + x)}$

$= \frac{1}{r}\int \operatorname{cosec}(\varphi + x)dx = \frac{1}{r}\log\left(\tan\left(\frac{x}{2} + \frac{\varphi}{2}\right)\right)$

$= \frac{1}{\sqrt{a^2 + b^2}}\log\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{a}{b}\right)$

Q.32 Find the area common to the circles $x^2 + y^2 - 2ax = 0$ and $x^2 + y^2 - 2ay = 0$. **(10)**

Ans:

Given circles $x^2 + y^2 - 2ax = 0$ and $x^2 + y^2 - 2ay = 0$

intersect at (0, 0) and (a, a)

Common area = $\int_0^a [y_1 - y_2]dx$ where $x_1^2 + y_1^2 - 2ax_1 = 0$ and

$= \int_0^a \left[\sqrt{a^2-(x-a)^2} - \left\{\sqrt{a^2-x^2}+a\right\}\right]dx$ $x_2^2 + y_2^2 - 2ay_2 = 0$

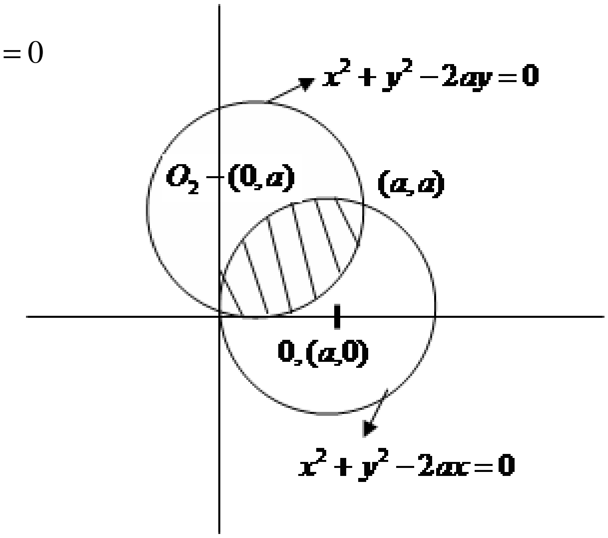
$= \int_0^a \sqrt{a^2-(x-a)^2}\,dx - \int_0^a \sqrt{a^2-x^2}\,dx - \int_0^a a\,dx$

Put x – a = t

$\int_0^a \sqrt{a^2-t^2}\,dt - \int_0^a \sqrt{a^2-x^2}\,dx - a^2$

$= \int_0^a \sqrt{a^2-z^2}\,dz - \int_0^a \sqrt{a^2-x^2}\,dx - a^2 = -a^2$

Hence required common area = a^2.



Q.33 Evaluate $\int_0^1 \frac{x^3}{(1+x^8)}\,dx$. **(6)**

Ans:

$\int_0^1 \frac{x^3 dx}{1+x^8}$ put $x^4 = 1$ $\quad \therefore x^3 dx = \frac{1}{4}dt$

$= \int_0^1 \frac{1}{4}\frac{dt}{1+t^2} = \frac{1}{4}[\tan^{-1} t]_0^1 = \frac{1}{4}.\frac{\pi}{4} = \frac{\pi}{16}$

Q.34 Solve following the differential equations

(i) $ydx - xdy = \sqrt{(x^2+y^2)}dx$. **(8)**

(ii) $\cos^2 x \frac{dy}{dx} + y = \tan x$. **(8)**

Ans:

(i) $\left(y - \sqrt{x^2+y^2}\right)dx = xdy$

$ydx - xdy = \sqrt{(x^2+y^2)}dx$

Or $\frac{dy}{dx} = \frac{y - \sqrt{x^2+y^2}}{x}$

Put y = vx, $\frac{dy}{dx} = v + \frac{xdv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{vx - \sqrt{x^2+v^2x^2}}{x} = v - \sqrt{1+v^2}$

$\therefore \frac{dv}{\sqrt{1+v^2}} + \frac{dx}{x} = 0$

Integrating, $\log\left(v+\sqrt{1+v^2}\right)+\log x = const$

Or $x\left[v+\sqrt{1+v^2}\right]=const$

$y+\sqrt{x^2+y^2}=const$.

(ii) $\cos^2 x\dfrac{dy}{dx}+y=\tan x$

Or $\dfrac{dy}{dx}+\dfrac{1}{\cos^2 x}y=\dfrac{\tan x}{\cos^2 x}=\tan x\sec^2 x$

It is linear differential equation with

I.F = $e^{\int\frac{1}{\cos^2 x}dx}=e^{\tan x}$

$\therefore$ Solution is $y.e^{\tan x}=\int e^{\tan x}\tan x\sec^2 xdx$

$= \tan xe^{\tan x}-\int e^{\tan x}.\sec^2 xdx+c$

$= \tan xe^{\tan x}-e^{\tan x}+c$

Hence required solution is

$ye^{\tan x}=(\tan x-1)e^{\tan x}+c$

Q.35 Show that the sum to n terms of the series 1.3.5 + 3.5.7 + 5.7.9 + … is $n\left(2n^3+8n^2+7n-2\right)$. **(8)**

Ans: The r[th] term of the series is given by

$t_r = (2r-1)(2r+1)(2r+3)$

$= 8r^3+12r^2-2r-3$

$\therefore s_n$, the sum to n terms of the series is

$$s_n = 8\sum_{r=1}^{n} r^3+12\sum_{r=1}^{n} r^2-2\sum_{r=1}^{n} r-3n$$

$$=8\left[\frac{n(n+1)}{2}\right]^2+12\frac{n(n+1)(2n+1)}{6}-2\frac{n(n+1)}{2}-3n$$

$$= n(2n^3+8n^2+7n-2)$$

Q.36 If α,β are the roots of the quadratic equation $x^2+px+1=0$ and γ,δ are the roots of the quadratic equation $x^2+qx+1=0$, then show that $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)=q^2-p^2$. **(8)**

Ans:

We have $\alpha+\beta=-p$, $\alpha\beta=1$

$\gamma+\delta=-q$ $\gamma\delta=1$

Now $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$

$=\left[\alpha\beta-(\alpha+\beta)\gamma+\gamma^2\right]\left[\alpha\beta+(\alpha+\beta)\delta+\delta^2\right]$

$=(\gamma^2+p\gamma+1)(\delta^2-p\delta+1)$

As γ, δ are roots of $x^2 + qx + 1 = 0$,

$\gamma^2 + 1 = -q\gamma$ and $\delta^2 + 1 = -q\delta$

Therefore,

$$(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = (-q\gamma + p\gamma)(-q\gamma - p\gamma)$$

$$= (q^2 - p^2)\gamma\delta = q^2 - p^2.$$

(since $\gamma\delta = 1$)

Q.37 If A + B + C = 180°, prove that $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A \sin B \sin C$. **(8)**

Ans:

We have B+C-A = 180 – 2A. so that

L.H.S. = sin (180 – 2A) + sin(180 – 2B) + sin(180 – 2C)

= sin 2A + sin 2B + sin 2C

= 2 sinA cosA + 2 sin(B + C) cos (B - C)

= -2 sinA [cos (B + C)] + 2 sinA cos(B - C)

= 2 sinA [cos(B – C) – cos(B + C)]

= 2 sinA 2sinB sinC

= 4 sinA sinB sinC

Q.38 Show that $\sin\frac{\pi}{14}$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$. **(8)**

Ans: It is sufficient to show that

$$8\sin^3\frac{\pi}{14} - 4\sin^2\frac{\pi}{14} - 4\sin\frac{\pi}{14} + 1 = 0$$

$$\text{L.H.S.} = 8\sin^3\frac{\pi}{14} - 4\left(\frac{1 - \cos\frac{\pi}{7}}{2}\right) - 4\sin\frac{\pi}{14} + 1$$

$$\because \quad \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$= 8\sin^3\frac{\pi}{14} - 4\sin\frac{\pi}{14} + 2\cos\frac{\pi}{7} - 1$$

$$= 4\sin\frac{\pi}{14}\left(2\sin^2\frac{\pi}{14} - 1\right) + 2\cos\frac{\pi}{7} - 1$$

$$= -4\sin\frac{\pi}{14}\cos\frac{\pi}{7} + 2\cos\frac{\pi}{7} - 1$$

$$= 4\cos\left(\frac{\pi}{2}+\frac{\pi}{14}\right)\cos\frac{\pi}{7} + 2\cos\frac{\pi}{7} - 1$$

$$= 4\cos\frac{8\pi}{14}\cos\frac{\pi}{7} + 2\cos\frac{\pi}{7} - 1$$

$$= 2\left(\cos\frac{5\pi}{7} + \cos\frac{3\pi}{7}\right) + 2\cos\frac{5\pi}{7} - 1$$

$$= \frac{1}{\sin\frac{\pi}{7}}\left[2\sin\frac{\pi}{7}\cos\frac{5\pi}{7} + 2\cos\frac{3\pi}{7}.\sin\frac{\pi}{7} + 2\cos\frac{\pi}{7}.\sin\frac{\pi}{7}\right] - 1$$

$$= \frac{1}{\sin\frac{\pi}{7}}\left[\left(\sin\frac{6\pi}{7} - \sin\frac{4\pi}{7}\right) + \left(\sin\frac{4\pi}{7} - \sin\frac{2\pi}{7}\right) + \sin\frac{2\pi}{7}\right] - 1$$

$$= \frac{\sin\frac{6\pi}{7}}{\sin\frac{\pi}{7}} - 1 \quad = \frac{\sin\left(\pi - \frac{\pi}{7}\right)}{\sin\frac{\pi}{7}} - 1 \quad = 0$$

Q.39 Find the value of c_1 such that the circles $x^2 + y^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 + 2x + 2y + c_1 = 0$ touch each other. **(8)**

Ans:
$x^2 + y^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 + 2x + 2y + c_1 = 0$ touch each other if the distance between their centre's is equal to the sum or difference of their radii.
centres of circles is (-1, -1) (-1, -1)
r^1 = radius is $\sqrt{1+1-1} = 1$
$r^2 = \sqrt{1+1-c_1} = \sqrt{2-c_1}$
distance between centres is 0
$\therefore \left|\sqrt{2-c_1} \pm 1\right| = 0$
i.e. $2 - c_1 + 1 \pm 2\sqrt{2 - c_1} = 0$
$\Rightarrow \quad c_1 = 1$

Q.40 For what values of k the points $(-1,4),(2,-2)$ and $(-4-k, 6-2k)$ are collinear?

Ans: **(8)**
The points (-1, 4) (2, -2) and (-4 - k, 6 – 2k) are collinear iff

$$\begin{vmatrix} -1 & 4 & 1 \\ 2 & -2 & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0$$

$R_1 \rightarrow R_1 - R_2$

$R_2 \rightarrow R_2 - R_3$

$$\begin{vmatrix} -3 & 6 & 1 \\ 6+k & -8+2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0$$

$\Rightarrow -3(2k - 8) - 6(6 + k) = 0$

$\Rightarrow -12k - 12 = 0$

$\Rightarrow k = -1$

Q.41 Find the equation of the circle for which $x - y - 1 = 0$ is a tangent and $x + y = 0$, $x - y + 4 = 0$ are normals. **(8)**

Ans:
Any two normals of a circle intersect at the centre of the circle. So, the centre is obtained by solving the equations of normals.
The point of intersection of the normals x + y = 0 and x – y + 4 = 0 is the point(-2, 2)

Now, the radius of the circle is the perpendicular distance from the centre of the circle to any target.
Hence,
Radius = perpendicular distance from (-2,2) to the target x –y – 1 = 0

$$= \left| \frac{-2-2-1}{\sqrt{1^2 + (-1)^2}} \right| = \frac{5}{\sqrt{2}}$$

Hence, the centre of the circle is (-1, 3) and the radius is $\frac{5}{\sqrt{2}}$.

So, the equation of the circle is

$$(x-(-2))^2 + (y-2)^2 = \left(\frac{5}{\sqrt{2}}\right)^2$$

$$or, \quad x^2 + 4x + 4 + y^2 - 4y + 4 = \frac{25}{2}$$

$$or, \quad 2x^2 + 2y^2 + 8x - 8y - 9 = 0$$

Q.42 Find the values of a, b such that the line ax + by + 1 = 0 is tangent to the hyperbola $3x^2 - y^2 = 3$ and is parallel to the line y = 2x + 4. **(8)**

Ans:
The equation of the hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1 \quad \text{or,} \quad \frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1,$$

where $\alpha^2 = 1$ and $\beta^2 = 3$.

The straight line y = mx + c is a target to the hyperbola

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1 \quad \text{if} \quad c^2=\alpha^2 m^2-\beta^2.$$

Since the straight line ax + by+1 = 0 is parallel to the line y = 2x + 4, thus m = 2.

$$\textit{Hence, } c^2=\alpha^2 m^2-\beta^2$$
$$=1.4-3=1$$
$$\Rightarrow c=\pm 1$$

Substituting in y = mx + c, we get y = 2x ± 1.

Thus, the required straight lines are y = 2x + 1 and y = 2x - 1

Or, 2x – y + 1 = 0 and –2x + y + 1 = 0

Hence, the values of a and b are: a = 2, b = -1 and a = -2, b = 1.

Q.43 Evaluate the limit $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$. **(8)**

Ans:

$$\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$=\lim_{x\to 0}\frac{\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{1+x^2}}{3x^2} \qquad [\text{L'Hospital rule}]$$

$$=\frac{1}{3}\lim_{x\to 0}\frac{1}{x^2}\left[\frac{1+x^2-\sqrt{1-x^2}}{(1+x^2)\sqrt{1-x^2}}\right]$$

$$=\frac{1}{3}\lim_{x\to 0}\frac{1}{x^2}\,\frac{(1+x^2)^2-(1-x^2)}{(1+x^2)\sqrt{1-x^2}}.\frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

[*Multiplying* $(1+x^2)+\sqrt{1-x^2}$ *to the numerator and denominator]*

$$=\frac{1}{3}\lim_{x\to 0}\frac{x^2(3+x^2)}{x^2(1+x^2)}.\frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

$$=\frac{1}{3}.3.\frac{1}{2}=\frac{1}{2}.$$

Q.44 Consider the function $f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & \text{for } x\neq 0,\\ 0, & \text{for } x=0.\end{cases}$. Find $f'(0)$ using first principle. Is $f'(x)$ continuous at $x=0$? **(8)**

Ans:

$$f'(0) = \lim_{x\to 0} \frac{f(h)-f(0)}{h} = \lim_{x\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{x\to 0} \frac{\sin\frac{1}{h}}{\frac{1}{h}} = 0.$$

If $x \neq 0$, then $f'(x) = 2x \sin\frac{1}{x} + x^2 (\cos\frac{1}{x})\left(-\frac{1}{x^2}\right)$

$$= 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

Now, $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right).$

we write $\cos\frac{1}{x} = 2x \sin\frac{1}{x} - \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right).$

Now, $\lim_{x\to 0} 2x \sin\frac{1}{x} = 0$, so that if $\lim_{x\to 0} f'(x)$ exists, then

$\lim_{x\to 0} \cos\frac{1}{x}$ will also exist, which is not true.

Hence, $\lim_{x\to 0} f'(x)$ does not exist.

that is, f' is not continuous at $x = 0$.

Q.45 Find the local maximum and minimum values of $f(x) = e^{\sin x}$ in $(0,2\pi)$. **(8)**

Ans:

Let $f(x) = e^{\sin x}$ then $f'(x) = \cos x\, e^{\sin x}$

$f'(x) = 0 \Rightarrow \cos x = 0$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{since } x \in [0,2\pi].$$

Now $f''(x) = (\cos^2 x - \sin x)e^{\sin x}$

If $x = \pi/2$, then $f''(\pi/2) = (0 - 1)e' = - e < 0$

So, $x = \pi/2$ is a point of maximum

If $x = 3\pi/2$, then $f''(x) = f''(3\pi/2) = (0 - (-1))e^{-1} = e^{-1} > 0.$

So, $x = 3\pi/2$ is a point of minimum

Hence, the maximum value of f is $f\left(\frac{\pi}{2}\right) = e$

the minimum value of f is $f\left(\frac{3\pi}{2}\right) = e^{-1}.$

Q.46 Find the area of the region bounded by $y = x^2 + 2$, $y = -x, x = 0$ and $x = 1$. **(8)**

Ans:

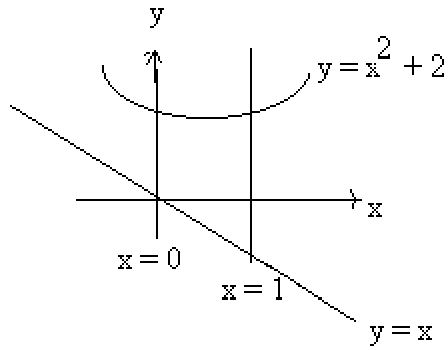
let f(x) = -x and g(x) = x^2 + 2.

Then f(x) ≤ g(x) and x in [0, 1]. Hence the required area is

$$\int_0^1 [g(x) - f(x)]\,dx$$

$$= \int_0^1 [x^2 + 2 + x]\,dx$$

$$= \left|\frac{x^3}{3} + 2x + \frac{x^2}{2}\right|_0^1 = \frac{17}{6}.$$



Q.47 Evaluate the following integral $\int \frac{dx}{\cos^6 x + \sin^6 x}$. **(8)**

Ans:

$$\int \frac{dx}{\cos^6 x + \sin^6 x}$$

$$= \int \frac{dx}{(\cos^2 x + \sin^2 x)(\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x)}$$

$$= \int \frac{dx}{1 - 3\sin^2 x \cos^2 x} = \int \frac{(1+\tan^2 x)\sec^2 x}{(1+\tan^2 x)^2 - 3\tan^2 x}$$

(multiplying and dividing by $\cos^4 x$)

$$= \int \frac{1+t^2}{(1+t^2)^2 - 3t^2}\,dt \qquad \text{(putting } \tan x = t\text{)}$$

$$= \int \frac{1+t^2}{t^4 - t^2 + 1}\,dt \quad = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} - 1}\,dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t^2}\right)^2 + 1}\,dt = \tan^{-1}\left(t - \frac{1}{t}\right) + c$$

$$= \tan^{-1}(\tan x - \cot x) + c.$$

Q.48 Evaluate the following definite integral $\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$. **(8)**

Ans:

$$\frac{x^3}{x^2+2x+1} = \frac{x^3}{(x+1)^2} = (x-2) + \frac{3x+2}{(x+1)^2}.$$

$$Now, \frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow A(x+1) + B = 3x+2$$

$\Rightarrow A = 3, B = -1.$

Thus,

$$\int_0^1 \frac{x^3}{x^2+2x+1}dx = \int_0^1 (x-2)dx + \int_0^1 \frac{3(x+1)-1}{(x+1)^2}dx$$

$$= \int_0^1 (x-2)dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2}$$

$$= \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$$

$$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) \qquad \cdots\cdots 1$$

$$= 3\ln 2 - 2.$$

Q.49 Solve the differential equation $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$. **(8)**

Ans:

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$$

An integrating factor is

$$\exp\left(\int\left(\frac{2x+1}{x}\right)dx\right) = \exp(2x + \ln|x|) = x\,e^{2x}.$$

multiplying the given equation through by $x\,e^{2x}$, we get

$$xe^{2x}\frac{dy}{dx} + e^{2x}(2x+1)y = x$$

or, $\frac{d}{dx}\left(xe^{2x}y\right) = x$

integrating, we get

$$x\,e^{2x}y = \frac{x^2}{2} + c.$$

Hence, the solution is

$$y = \frac{1}{2}xe^{-2x} + \frac{c}{x}e^{-2x}$$

where c is an arbitrary constant.

Q.50 Solve the differential equation

$$x \sin y dx + \left(x^2 + 1\right)\cos y dy = 0 .$$ **(8)**

Ans:

Separating the variables by dividing by $(x^2+1)\sin y$, we get

$$\frac{x}{x^2+1}dx + \frac{\cos y}{\sin y}dy = 0$$

Thus, $\int \frac{x}{x^2+1}dx + \int \frac{\cos y}{\sin y}dy = c$

or, $\frac{1}{2}\ln(x^2+1) + \ln|\sin y| = c.$

or, $\ln(x^2+1) + 2\ln|\sin y| = 2c$

or, $\ln[(x^2+1)\sin^2 y] = 2c = \ln k$, say

or, $(x^2+1)\sin^2 y = k,$ ------- (*)

where k is an arbitrary constant.

In dividing by $(x^2 + 1)\sin y$, we assumed that $\sin y \neq 0$.

now, consider $\sin y = 0$. These are given by $y = n\pi$, $n = 0, \pm1, \pm2, \ldots$ Writing the original differential equation in the derivative form, it is clear that $y = n\pi$ is a constant solution. Each of these constant solution is present in the solution (*). So, we have not lost any solution in the division process.

Q.51 Show that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1+x)^{2n-1}$. **(8)**

Ans:

coefficient of x^n in expansion of $(1 + x)^{2n} = {}^{2n}C_n$ ....(1)

coefficient of x^n in expansion of $(1 + x)^{2n-1} = {}^{2n-1}C_n$ ....(2)

Thus ratio of (1) to (2) = $\frac{\lfloor 2n}{\lfloor 2n-n \lfloor n} \frac{\lfloor 2n-1-n \lfloor n}{\lfloor 2n-1} = \frac{2n\lfloor n-1}{\lfloor n} = \frac{2n}{n} = 2$

Thus coefficient of x^n is expansion of $(1+x)^{2n}$ is double the coefficient of x^n in $(1+x)^{2n-1}$.

Q.52 If $x = 1 + a + a^2 + ...\infty$ and $y = 1 + b + b^2 + ...\infty$, where $|a| < 1,\ |b| < 1$ then prove that $1 + ab + a^2b^2 + a^3b^3.....\infty = \dfrac{xy}{x + y - 1}$. **(8)**

Ans:

$x = 1 + a + a^2 + a^3 + ...\infty = \dfrac{1}{1-a}$ (sum of an infinite G.P.)

or $x - ax = 1$ or $a = \dfrac{x-1}{x}$

Similarly $b = \dfrac{y-1}{y}$

$\therefore\ 1 + ab + a^2b^2 + a^3b^3 + ...\infty = \dfrac{1}{1-ab}$ (sum of an infinite G.P.)

$= \dfrac{1}{1 - \dfrac{x-1}{x}.\dfrac{y-1}{y}} = \dfrac{xy}{x+y-1}$

Q.53 If $A + B + C = \pi$, show that
$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$. **(8)**

Ans:

LHS = sin 2A + sin 2B + sin 2C

= 2 sin(A+B) cos (A-B) + 2 sinC cosC ($\therefore \sin c + \sin d = 2\sin\dfrac{c+d}{2}\cos\dfrac{c-d}{2}$

= 2 sin(π-c) cos(A-B) + 2sinC cos(A+B)

= 2 sinC cos(A-B) – 2 sinC cos(A+B)

= 2 sinC[cos(A-B) – cos(A+B)]

= 2 sinC[cosA cosB + sinA sinB – cosA cosB + sinA sinB]

= 4 sinA sinB sinC = R.H.S.

Q.54 If a, b, c be the sides opposite to the angles A, B, C of a triangle ABC, show that $\dfrac{b-c}{b+c} = \dfrac{\tan\dfrac{B-C}{2}}{\tan\dfrac{B+C}{2}}$. **(8)**

Ans:

LHS $= \dfrac{b-c}{b+c} = \dfrac{k\sin B - k\sin C}{k\sin B + k\sin C}$ $\left(\text{using sin formula } \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}\right)$

$= \dfrac{\sin B - \sin C}{\sin B + \sin C} = \dfrac{2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}}{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}} = \dfrac{\tan\dfrac{B-C}{2}}{\tan\dfrac{B+C}{2}} =$ R.H.S.

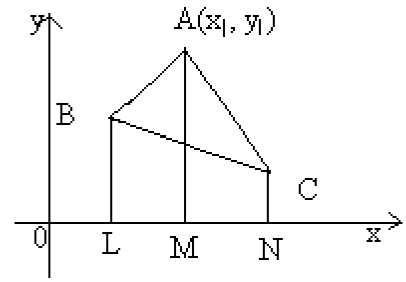
Q.55 Derive the formula for finding the area of a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$. **(8)**

Ans:

Drop $\perp_S$ AM, BL, CN from A, B, C on x-axis.

$\therefore$ Δ ABC = trap ABLM + trap AMNC - trap BLNC

$$= \frac{BL + AM}{2}.LM + \frac{AM + CN}{2}MN - \frac{BL + CN}{2}LN$$



$$= \frac{y_1 + y_2}{2}(x_1 - x_2) + \frac{y_1 + y_3}{2}(x_3 - x_1) - \frac{y_2 + y_3}{2}(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Q.56 Find the equation of a straight line joining the point (3, 5) to the point of intersection of the lines 4x +y = 1 and 7x – 3 y = 35. **(8)**

Ans:

Any line passing through the point of intersection of the given lines is

$4x + y - 1 + k(7x - 3y - 35) = 0$ ----------(1)

If (1) passes through (3, 5), $4\times3 + 5 - 1 + k(7\times3 - 3\times5 - 35) = 0$ or $k = \frac{16}{29}$

Thus required line $4x + y - 1 + \frac{16}{29}(7x - 3y - 35) = 0$ or $12x - y - 31 = 0$

Q.57 Find the equation of the circle which passes through the centre of the circle $x^2 + y^2 + 8x + 10y - 7 = 0$ and is concentric with the circle $2x^2 + 2y^2 - 8x - 12y - 9 = 0$. **(8)**

Ans:

Any circle concentrate with the given circle is

$2x^2 + 2y^2 - 8x - 12y + k = 0$ ------(1)

(1) passes through the centre of the circle $x^2+y^2-8x-10y+7=0$ with (-4,-5). Therefore (-4,-5) shall satisfy (1).

$\therefore\ 2\times16+2\times25-8(-4)-12(-5)+k=0$ or $k=174$

Thus required circle is

$2x^2+2y^2-8x-12y+174=0$

$\Rightarrow x^2+y^2-4x-6y+87=0$

Q.58 Find the focus, vertex, directrix and axis of the parabola $y=-4x^2+3x$. **(8)**

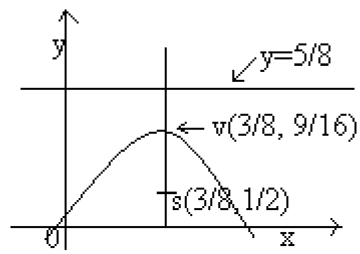
Ans:

The given parabola can be written as

$x^2-\frac{3}{4}x=-\frac{y}{4}$

or $\left(x-\frac{3}{8}\right)^2=-\frac{y}{4}+\frac{9}{64}$

$=-\frac{1}{4}\left(y-\frac{9}{16}\right)$....(1)



put $x-\frac{3}{8}=X,\ y-\frac{9}{16}=Y, \therefore$ (1) becomes $X^2=-\frac{1}{4}Y$

which represents a parabola with vertex at $\left(\frac{3}{8},\frac{9}{16}\right)$ and axis $x=\frac{3}{8}$

focus $S=\left(\frac{3}{8},\frac{9}{16}-\frac{1}{16}\right)$ i.e. $\left(\frac{3}{8},\frac{1}{2}\right)$

Direction $y=\frac{9}{16}+\frac{1}{16}=\frac{5}{8}$

The shape of the parabola is as shown in the figure

Q.59 Evaluate $\lim\limits_{x\to0}\frac{x\left(3^x-1\right)}{1-\cos x}$. **(8)**

Ans:

$$\lim_{x\to0}\frac{x(3^x-1)}{1-\cos x}=\lim_{x\to0}\frac{3^x-1}{x}.\frac{x^2}{1-\left(1-2\sin^2\frac{x}{2}\right)}=\log_e 3\lim_{x\to0}\frac{x^2}{2\sin^2\frac{x}{2}}$$

$$=\log_e 3\ \underset{\frac{x}{2}\to0}{lt}\ \frac{2}{\left(\frac{\sin x/2}{x/2}\right)^2}=2\log_e 3$$

Q.60 Find $\frac{dy}{dx}$, if $y = \sin^{-1}\frac{2\theta}{1+\theta^2}, x = \tan^{-1}\frac{2\theta}{1-\theta^2}$. **(8)**

Ans:
put $\theta = \tan\alpha$

$$\therefore y = \sin^{-1}\frac{2\tan\alpha}{1+\tan^2\alpha}, x = \tan^{-1}\frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$= \sin^{-1}\left(\frac{2\frac{\sin\alpha}{\cos\alpha}}{1+\frac{\sin^2\alpha}{\cos^2\alpha}}\right), x = \tan^{-1}(\tan(\alpha+\alpha))$$

$$y = \sin^{-1}(2\sin\alpha\cos\alpha), x = 2\alpha$$

$$y = 2\alpha \quad , \quad x = 2\alpha$$

$$\frac{dy}{dx} = \frac{dy}{d\alpha}.\frac{d\alpha}{dx} = 2\times\frac{1}{2} = 1$$

Q.61 Derive the equation of the tangent and the normal to the curve $y^2 = 4ax$ at the point $(at^2, 2at)$. **(8)**

Ans:

$$y^2 = 4ax, \therefore 2y\frac{dy}{dx} = 4a \text{ or } \frac{dy}{dx} = \frac{4a}{2\sqrt{4ax}} = \sqrt{\frac{a}{x}} = \frac{1}{t} \text{ at}(at^2, 2at)$$

$\therefore$ Equation of the tangent at$(at^2, 2at)$ is

$$y - 2at = \frac{dy}{dx}(x - at^2)$$

$$y - 2at = \frac{1}{t}(x - at^2) \text{ or } yt = x + at^2$$

Q.62 Evaluate $\int \frac{x+\sin x}{1+\cos x}dx$. **(8)**

Ans:

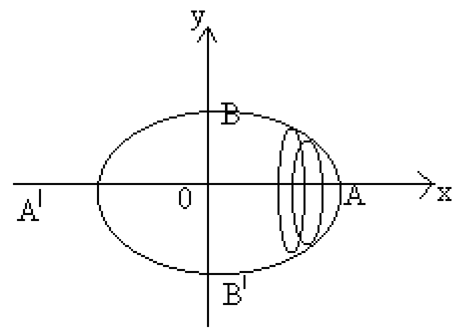
$$\int\frac{x+\sin x}{1+\cos x}dx = \int\frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}dx$$

$$= \int\frac{1}{2}x\sec^2\frac{x}{2}dx + \int\tan\frac{x}{2}\quad dx$$

$$= x\tan\frac{x}{2} - \int 1.\tan\frac{x}{2}dx + \int\tan\frac{x}{2}dx = x\tan\frac{x}{2}$$

Q.63 Find the volume of the solid of revolution obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about x-axis. **(8)**

Ans:



Required volume of revolution $=2\int_0^a \pi y^2 dx$

$$=2\pi\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$=2\pi b^2 a-2\pi\frac{b^2}{a^2}\int_0^a x^2 dx$$

$$=2\pi b^2 a-2\pi\frac{b^2}{a^2}.\frac{a^3}{3}=\frac{4}{3}\pi ab^2$$

Q.64 Evaluate $\int_0^{\frac{\pi}{2}} \sin^n x dx$, for any positive integer n. **(8)**

Ans:

$$I_n=\int_0^{\pi/2}\sin^n x dx=\int_0^{\pi/2}\sin^{n-1}x\sin x dx=-\sin^{n-1}x\cos x\Big|_0^{\pi/2}+\int_0^{\pi/2}(n-1)\sin^{n-2}x\cos^2 x \quad dx$$

$$=(n-1)\int_0^{\pi/2}\sin^{n-2}x(1-\sin^2 x)dx=(n-1)I_{n-2}-(n-1)I_n$$

or changing sides

$$nI_n=(n-1)I_{n-1} \text{ or } I_n=\int_0^{\pi/2}\sin^n x dx=\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x \quad dx$$

continuing the process

$$I_n=\frac{n-1}{n}I_{n-2}=\frac{n-1}{n}\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3}I, \quad \text{if n is odd}$$

$$=\frac{n-1}{n}\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{1}{2}I_0, \quad \text{if n is even}$$

where $I_1 = \int_0^{\pi/2} \sin x dx = 1$

$$I_0 = \int_0^{\pi/2} x dx = \frac{\pi}{2}$$

Hence $\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n}\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3}$, if n is odd

$$= \frac{n-1}{n}\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{1}{2}.\frac{\pi}{2}, \quad \text{if } n \text{ is even}$$

Q.65 **(i)** $\frac{dy}{dx} = e^{3x-y} + x^2 e^{-y}$.

(ii) $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$.

(iii) $\left(1 + x^2\right)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$. **(16)**

Ans: (i)

$$\frac{dy}{dx} = e^{3x-y} + x^2 e^{-y} = (e^{3x} + x^2)e^{-y}$$

or $e^y dy = (e^{3x} + x^2)dx$

integrating, $\frac{1}{3}e^{3x} + \frac{x^3}{3} + c = e^y$

$\Rightarrow y = \log(1/3e^{3x} + x^3/3 + C)$

(ii)

Given equation can be rewritten as $\frac{dy}{dx} = \frac{y-x}{y+x}$ ....(1)

put $y = 1/x, \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes $v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v-1}{v+1}$

or $x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{-1-v^2}{v+1} = -\frac{1+v^2}{v+1}$

or $\left(\frac{v}{v^2+1} + \frac{1}{v^2+1}\right)dv = -\frac{dx}{x}$

Integrating, $\frac{1}{2}\log(1+v^2)+\tan^{-1}v=-\log x+c$

or $\frac{1}{2}\log\left(1+\frac{y^2}{x^2}\right)+\tan^{-1}\frac{y}{x}+\log x=\text{const}$

or $\frac{1}{2}\log(x^2+y^2)+\tan^{-1}\frac{y}{x}=\text{const}$

(iii)

Given equation can be rewritten as

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y-\frac{1}{(1+x^2)^2}$$

it's I.F. is $=e^{\int\frac{2x}{1+x^2}dx}=e^{\log(1+x)^2}=1+x^2$

Hence solution is

$$y(1+x^2)=\int(1+x^2)\frac{1}{(1+x^2)^2}dx+c=\int\frac{1}{1+x^2}dx+c$$

$$y(1+x^2)=\tan^{-1}x+c$$

Q.66 The sum of first p terms of an A.P. is the same as the sum of its first q terms. Find the sum of its first (p + q) terms. **(8)**

Ans:

$$\frac{p}{2}[2a+(p-1)d]=\frac{q}{2}[2a+(q-1)d]$$

$$\therefore\frac{p}{q}=\frac{2a+(q-1)d}{2a+(p-1)d}$$

$$\frac{p+q}{p-q}=\frac{4a+(p+q-2)d}{(q-1-p+1)d}$$

Or $\frac{p+q}{p-q}=\frac{4a+(p+q-2)d}{(q-p)d}$

$\therefore -(p+q)d=4a+(p+q-2)d$

Or 4a + (p + q – 2)d + (p + q)d = 0

Or 4a + (2p + 2q – 2)d = 0

Or 2[2a + (p+ q – 1)d] = 0

Or $\frac{p+q}{2}[2a+(p+q-1)d]=0$

Thus sum of (p + q) terms is 0.

Q.67 For what value of n are the coefficients of second, third and fourth terms in the expansion of $(1+x)^n$ in A.P.? **(8)**

Ans:

Since 2^{nd}, 3^{rd}, 4^{th} terms of $(1 + x)^n$ are in A.P. Thus

$^{n}C_1 + {}^{n}C_3 = 2\,{}^{n}C_2$

$\Rightarrow n + \frac{n(n-1)(n-2)}{3!} = \frac{2n(n-1)}{2!}$

Or $1 + (n-1)\left[\frac{n-2}{6} - 6\right] = 0$

$\Rightarrow 6 + (n-1)(n-8) = 0$

$\Rightarrow n^2 - 9n + 14 = 0 \qquad \Rightarrow n = 2,7$

n = 7 is only possible.

Q.68 Solve for θ the equation $\text{Sin } m\,\theta + \text{Sin } n\,\theta = 0$, where $m \neq n$. **(8)**

Ans:

$\sin mq + \sin nq = 0$

$\Rightarrow \sin\frac{(m+n)\theta}{2}.\cos\frac{(m-n)\theta}{2} = 0$

$\Rightarrow either \sin\frac{m+n}{2}\theta = 0$

$\Rightarrow \frac{(m+n)\theta}{2} = n\pi$

$\Rightarrow \theta = \frac{2n\pi}{m+n}$

And $\cos\left(\frac{(m-n)\theta}{2}\right) = 0$

$\Rightarrow \frac{(m-n)\theta}{2} = (2n+1)\frac{\pi}{2}$

$\Rightarrow \theta = \frac{(2n+1)\pi}{(m-n)}$

$\theta = \frac{2n\pi}{m+n} or \frac{(2n+1)\pi}{m-n}$

Q.69 If a, b, c be the sides opposite to the angles A, B, C for a triangle ABC, show that $\frac{a+b}{c} = \frac{\text{Cos}\frac{A-B}{2}}{\text{Sin}\frac{C}{2}}$. **(8)**

Ans:

$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}$

$= \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}$

$$= \frac{\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}$$

Q.70 Derive the formula for the angle between the straight lines $y = m_1x + c_1$ and $y = m_2x + c_2$. **(8)**

Ans:

Let $y = m_1x + C_1$ be the equation of line AC which makes an angle θ_1 with x – axis, so $m_1 = \tan\theta_1$ and $y = m_2x + C_2$ be the equation of line BC which makes an angle θ_2 with x –axis so $m_2 = \tan\theta_2$

The angle between the lines

$\theta = \theta_2 - \theta_1$ or $180 - (\theta_2 - \theta_1)$

$\therefore \tan\theta = \tan(\theta_2 - \theta_1)$

$\tan[180 - (\theta_2 - \theta_1)] = \tan(\theta_2 - \theta_1) or - \tan(\theta_2 - \theta_1)$

Thus $\tan\theta = -\tan(\theta_2 - \theta_1) or \tan(\theta_2 - \theta_1)$

$$= \pm\tan(\theta_2 - \theta_1)$$

$$= \pm\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2}$$

$$= \pm\frac{m_1 - m_2}{1 + m_1m_2}$$

$$\Rightarrow \therefore \tan\theta = \left|\frac{m_1 - m_2}{m_1m_2 + 1}\right|$$

$$\Rightarrow \theta = \tan^{-1}\left|\frac{m_1 - m_2}{m_1m_2 + 1}\right|$$

Q.71 Find the equation of a straight line which is perpendicular to 2x – 5y = 30 and the sum of its intercepts on the coordinate axes is 7. **(8)**

Ans:

Let the equation of the line is

y = mx + C

It is perpendicular to 5y = 2x – 30

$\therefore m.\frac{2}{5} = -1$

$\therefore m = \frac{-5}{2}$ $[m_1.m_2 = -1]$

$\therefore$ Equation of the line is

$y = \frac{-5}{2}x + c$

Or 2y + 5x = 2c

Its passing through (x, 0) therefore

$x = \frac{2c}{5}$

Again it is passing through (0, 7-x)

$\therefore 2(7-x) = 2c \qquad \Rightarrow c = 5$

Or 14 – 2x = 5 x 2 or x = 2

Equation of line is 2y + 5x = 10

Q.72 Find the equation of the circle concentric with the circle $2x^2 + 2y^2 + 8x + 10y - 39 = 0$ and having its area equal to 16π. **(8)**

Ans:

Centre of the circle $2x^2 + 2y^2 + 8x + 10y - 39 = 0$ is $\left(-2, \frac{-5}{2}\right)$

Or $x^2 + y^2 + 4x + 5y - \frac{39}{2} = 0$

Also area $\pi r^2 = 16\pi$

$\therefore r = 4$

Let the equation of the required circle is

$(x-h)^2 + (y-k)^2 = r^2$

$\Rightarrow (x+2)^2 + (y+5/2)^2 = 16$

$\Rightarrow x^2 + 4x + 4 + y^2 + 5y + \frac{25}{4} = 16$

$x^2 + y^2 + 4x + 5y - \frac{23}{4} = 0$

Q.73 Find the centre, eccentricity, foci and length of the latus rectum of the ellipse $4x^2 + 9y^2 - 8x + 36y + 4 = 0$. **(8)**

Ans:

$4x^2 + 9y^2 - 8x + 36y + 4 = 0$

$(4x^2 - 8x) + (9y^2 + 36y) + 4 = 0$

$4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) - 4 - 36 + 4 = 0$

$4(x-1)^2 + 9(y+2)^2 = 36$

$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$

Let x – 1 = X

y + 2 = Y,

thus $\frac{X^2}{9} + \frac{Y^2}{4} = 1$

Centre (0, 0) X = 0, Y = 0

i.e. (1, -2)

Eccentricity $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{9} = \frac{5}{9} \rightarrow e = \frac{\sqrt{5}}{3}$

Foci $X = \pm a \Rightarrow x - 1 = 3 \Rightarrow x = 4$
Or $x - 1 = -3 \Rightarrow x = -2$
Foci (4 , -2), (-2, -2)
Length = 4a = 4 x 3 = 12

Q.74 Differentiate from the first principle the function y = tan x. **(8)**

Ans:
y = tanx y = δy = tan(x + δx)

$$\lim_{\delta y \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\tan(x+\delta x) - \tan x}{\delta x} = \lim_{\delta x \to 0} \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\delta x.\cos x.\cos(x+\delta x)}$$

$$= \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}.\frac{1}{\cos x.\cos(x+\delta x)}$$

$$= \sec^2 x$$

Q.75 Evaluate $\lim\limits_{x \to 0} \frac{\sqrt{(1+x+x^2)} - 1}{\text{Sin}4x}$. **(8)**

Ans:

$$\lim_{x \to 0} \frac{\sqrt{(1+x+x^2)} - 1}{\sin 4x}$$

$$= \lim_{x \to 0} \frac{\frac{1+2x}{2\sqrt{1+x+x^2}}}{4\cos 4x} \qquad \text{(L-Hospital rule)}$$

$$= \frac{\frac{1}{2}(1+2x)}{4} = \frac{1}{8}$$

Q.76 Find the local maximum and minimum values of the function y = sin 3x – 3 sin x, $0 \le x < 2\pi$. **(8)**

Ans:
y = sin3x – 3sinx

$$\frac{dy}{dx} = 3\cos 3x - 3\cos x$$

$$\frac{dy}{dx} = 0 \Rightarrow \cos 3x - \cos x = 0$$

$$\Rightarrow 4\cos^3 x - 3\cos x - \cos x = 0$$

$$\Rightarrow 4\cos^3 x - 4\cos x = 0$$

$$\Rightarrow 4\cos x(\cos^2 x - 1) = 0$$

$$\Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2},$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

and $\cos^2 x - 1 = 0 \Rightarrow \cos x = 1 \Rightarrow \cos x = \cos 0$

$x = 0, 2\pi$

$$\frac{d^2y}{dx^2} = -9\sin 3x + 3\sin x$$

At $x = \frac{\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = 9 - 3 = 6 > 0$

$$x = \frac{3\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = -9\sin\frac{9\pi}{2} + 3\sin\frac{3\pi}{2} = -9 - 3 = -12 < 0$$

At x = 0, 2π, $\frac{d^2y}{dx^2} = 0$.

Thus maximum is obtained at $x = \frac{3\pi}{2}$ and maximum value is 4.

The minimum is obtained at $x = \frac{\pi}{2}$ and minimum value is -4.

Q.77 Evaluate $\int \frac{\mathrm{xdx}}{\sqrt{\mathrm{x}^2 + \mathrm{a}^2} + \sqrt{\mathrm{x}^2 - \mathrm{a}^2}}$. **(8)**

Ans:

$$\int \frac{xdx}{\sqrt{x^2 + a^2} + \sqrt{x^2 - a^2}}$$

$$= \frac{1}{2a^2}\int x\left(\sqrt{x^2 + a^2} - \sqrt{x^2 - a^2}\right)dx$$

$$= \frac{1}{4a^2}\int \sqrt{t}\,dt - \frac{1}{4a^2}\int \sqrt{u}\,du$$

Let $x^2 + a^2 = t \quad xdx = \frac{dt}{2}$

Also let $x^2 - a^2 = u \quad xdx = \frac{du}{2}$

$$= \frac{1}{4a^2}\left(\frac{2}{3}t^{3/2} - \frac{2}{3}u^{3/2}\right)$$

$$= \frac{1}{6a^2}(x^2 + a^2)^{3/2} - \frac{1}{6a^2}(x^2 - a^2)^{3/2}$$

$$= \frac{1}{6a^2}\left\{(x^2 + a^2)^{3/2} - (x^2 - a^2)^{3/2}\right\}$$

Q.78 Find the area bounded by the curve $\sqrt{\mathrm{y}} + \sqrt{\mathrm{x}} = \sqrt{\mathrm{a}}$ and the coordinate axes. **(8)**

Ans:

Let x = 0 $\Rightarrow$ y = a

y = 0 $\Rightarrow$ x = a

$$\sqrt{y} = \sqrt{a} - \sqrt{x}$$

$$y = a - x - 2\sqrt{ax}$$

$$A = \int_a^0 y\,dx = \int \left(a - x - 2\sqrt{ax}\right)dx$$

$$= \left[ax + \frac{-x^2}{2} - 2\sqrt{a} \times \frac{2}{3} x^{3/2}\right]_a^0$$

$$= \left[-a^2 + \frac{a^2}{2} + \frac{4\sqrt{a}}{3} a^{3/2}\right]$$

$$= \left[\frac{-a^2}{2} + \frac{4a^2}{3}\right]$$

$$= \left[\frac{-3a^2 + 8a^2}{2 \times 3}\right] = \frac{5a^2}{6}$$

Q.79 Evaluate $\int_0^{\pi/2} \frac{\cos x\, dx}{(1 + \text{Sin } x)(2 + \text{Sin } x)}$. **(8)**

Ans:

$$I = \int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$$

$$= \int_1^2 \frac{dt}{t(t+1)}$$

$1 + \sin x = t$

$x = 0 \Rightarrow t = 1$

$x = \pi/2 \Rightarrow t = 2$

$$= \int_1^2 \frac{1}{t}dt - \int_1^2 \frac{1}{t+1}dt$$

$$= [\log t]_1^2 - [\log(t+1)]_1^2$$

$= \log 2 - \log 1 - \log 3 + \log 2$

$= 2\log 2 - \log 3$

$$= \log\frac{4}{3}$$

Q.80 Solve any **TWO** of the following differential equations:- **(24)**

(i) $xy\frac{dy}{dx} = 1 + x + y + xy$.

(ii) $\left(x^2 - y^2\right)dx = 2xy\,dy$.

(iii) $\left(1 - x^2\right)\frac{dy}{dx} - xy = 1$.

Ans:

(i)

$$xy\frac{dy}{dx}=1+x+y+xy$$

Or $xy\frac{dy}{dx}=(1+x)(1+y)$

Or $\frac{y}{(1+y)}dy=\frac{(1+x)}{x}dx$

Or $\frac{(1+y)-1}{(1+y)}dx=dx+\frac{dx}{x}$

Or $dy-\frac{1}{1+y}dy=dx+\frac{dx}{x}$

$\therefore\ y-\log(1+y)=x+\log x+c$

(ii) $(x^2-y^2)dx-2xy=0$

$M=x^2-y^2,\quad \frac{\partial M}{\partial y}=-2y$

$N=-2xy,\quad \frac{\partial N}{\partial x}=-2y$

Thus $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, therefore eq is exact. Hence solution is

$\int(x^2-y^2)dx=C$

Or $\frac{x^3}{3}-xy^2=C$.

(iii)

$$(1-x^2)\frac{dy}{dx}-xy=1$$

Or $\frac{dy}{dx}-\frac{x}{1-x^2}y=\frac{1}{1-x^2}$

I.F = $e^{-\int\frac{x}{1-x^2}dx}$

$= e^{\frac{1}{2}\int\frac{-2x}{1-x^2}dx}$

$= e^{\frac{1}{2}\log(1-x^2)}$

$= \sqrt{1-x^2}$

$\therefore\ y\sqrt{1-x^2}=\int\frac{1}{\sqrt{1-x^2}}dx+c$

$\therefore\ y\sqrt{1-x^2}=\sin^{-1}x+c$.

Q.81 If 5 times the 5^{th} term of an A.P. is equal to the 10 times the 10^{th} term, find the 15^{th} term of the A.P. **(8)**

Ans:

5^{th} term of an A.P = a + 4d

10^{th} term of an A.P = a + 9d

Here 5(a + 4d) = 10(a + 9d)

a + 4d = 2a + 18d

a = -14d

$t_{15} = a + 14d$

= a – a = 0

Q.82 If S_n denotes the sum of n terms of a G.P., prove that $(S_{10} - S_{20})^2 = S_{10}(S_{30} - S_{20})$. **(8)**

Ans:

$$s_{10}(s_{30} - s_{20}) = \frac{a(1-r^{10})}{1-r}\left[\frac{a(1-r^{30})}{1-r} - \frac{a(1-r^{20})}{1-r}\right]$$

$$= \frac{a^2(1-r^{10})}{(1-r)^2}\left[1 - r^{30} - 1 + r^{20}\right]$$

$$= \frac{a^2(1-r^{10})(r^{20} - r^{30})}{(1-r)^2}$$

$$= \frac{a^2 r^{20}(1-r^{10})^2}{(1-r)^2}$$

$$(s_{10} - s_{20})^2 = \left[\frac{a(1-r^{10})}{1-r} - \frac{a(1-r^{20})}{1-r}\right]^2$$

$$= \frac{a^2}{(1-r)^2}\left[1 - r^{10} - 1 + r^{20}\right]^2$$

$$= \frac{a^2 r^{20}}{(1-r)^2}\left[r^{10} - 1\right]^2$$

$$= \frac{a^2 r^{20}}{(1-r)^2}\left[1 - r^{10}\right]^2$$

L.H.S = R.H.S

Q.83 Show that $\frac{\sin A - \sin 3A + \sin 5A - \sin 7A}{\cos A - \cos 3A - \cos 5A + \cos 7A} = \cot 2A$. **(8)**

Ans:

$$\text{L.H.S} = \frac{\sin a - \sin 3a + \sin 5a - \sin 7a}{\cos a - \cos 3a - \cos 5a + \cos 7a}$$

$$= \frac{-2\cos 4a \sin 3a + 2\cos 4a \sin a}{2\cos 4a \cos 3a - 2\cos 4a \cos a}$$

$$= \frac{\sin a - \sin 3a}{\cos 3a - \cos a}$$

$$= \frac{-2\cos 2a \sin a}{-2\sin 2a \sin a}$$

= cot 2a = R.H.S

Q.84 If in the triangle ABC, A = 60°, prove that $\frac{1}{c+a}+\frac{1}{a+b}=\frac{3}{a+b+c}$. **(8)**

Ans:
To prove

$$\frac{1}{c+a}+\frac{1}{c+b}=\frac{3}{a+b+c}$$

$$\Rightarrow (2a+b+c)(a+b+c)=3(a+c)(a+b)$$

$$\Rightarrow 2a^2+2ab+2ac+ab+b^2+bc+ac+bc+c^2$$
$$=3(a^2+ab+ca+cb)$$

$$\Rightarrow b^2+c^2=bc+a^2$$

$$\Rightarrow \frac{b^2+c^2-a^2}{2bc}=\frac{1}{2}$$

$$\Rightarrow \cos A=\frac{1}{2}$$

This is true since $A=60^0$.

Q.85 Find the equation of the straight line which passes through the intersection of the lines x + y – 3 = 0 and 2x – y = 0 and is inclined at an angle of 45° with x-axis. **(8)**

Ans:
Point of intersection is (1, 2)
x + y = 3

$$\frac{2x-y=0}{3x=3},\qquad x = 1,\ y = 2$$

Let the equation of the line is
y = wx + c
Here w = tan 45^0 = 1.
And became the line passing through (1, 2) therefore
2 = 1 + C $\therefore C=1$
Therefore the equation of required line is y = x + 1 i.e. x – y + 1 = 0

Q.86 Show that $9x^2+4y^2-54x-56y+241=0$ represents an ellipse. Find its centre, vertices, foci, eccentricity, directrices, latusrectum and equations of major and minor axis. **(8)**

Ans:

$$9x^2+4y^2-54y-56y+241=0$$

$$\Rightarrow (9x^2-54x)+(4y^2-56y)+241=0$$

$$\Rightarrow 9(x^2-6x+9)+4(y^2-14y)+241=0$$

$$\Rightarrow 9(x^2-6x+9)+4(y^2-14y+49)+241-81-196=0$$

$$\Rightarrow 9(x-3)^2+4(y-7)^2-277+241=0$$

$$\Rightarrow 9(x-3)^2+4(y-7)^2-36=0$$

$$\Rightarrow \frac{(x-3)^2}{4}+\frac{(y-7)^2}{9}=1$$

Let x – 3 = X, y – 7 = Y

$\Rightarrow \frac{X^2}{4}+\frac{Y^2}{9}=1$ …………….(1)

Center of the ellipse = (0, 0)

X = 0 $\Rightarrow$ x – 3 = 0 $\Rightarrow$ x = 3, Y = 0 $\Rightarrow$ y – 7 = 0 $\Rightarrow$ y = 7

Center = (3, 7)

About major axis: - x = a $\Rightarrow$ x – 3 = 2 $\Rightarrow$ x = 5; x = 5. Also x = -a $\Rightarrow$ x – 3 = 2 $\Rightarrow$ x = (5, 7)(1, 7)

Y = 0 $\Rightarrow$ y – 7 = 0 $\Rightarrow$ y = 7

About minor axis: - X = 0 $\Rightarrow$ x – 3 = 0 $\Rightarrow$ x = 3

Y = b $\Rightarrow$ y – 7 = 3 $\Rightarrow$ y = 10

Y = -b $\Rightarrow$ y – 7 = -3 $\Rightarrow$ y = 4

For y = $\pm$ b $\Rightarrow$ y – 7 = 3 $\Rightarrow$ y = 10,

y – 7 = 3 $\Rightarrow$ y = 4

foci (3, 10)(3, 4)

Eccentricity

$$e^2 = 1-\frac{a^2}{b^2} \qquad = 1-\frac{4}{9}=\frac{5}{9}$$

$$e=\left(\frac{\sqrt{5}}{3}\right)$$

Directories y = $\pm$ b $\Rightarrow$ y – 7 = $\pm$ 3 $\Rightarrow$ y – 10 = 0, y – 4 = 0

Latus rectum 4a = 4 x 2 = 8

Equation x = a $\Rightarrow$ x = 5

X = -a $\Rightarrow$ x = 1

Minor Axis y = b $\Rightarrow$ y = 10

y = -b $\Rightarrow$ y = 4

Q.87 Find the equation of the circle which passes (4, 1) & (6, 5) and having centre on the line 4x+y =16. **(8)**

Ans:

Equation of the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ …………….(*)

Which passes through (4, 1) and (6, 5)

16 + 1 + 8g + 2f + c = 0

8g + 2f + c + 17 = 0 ………………………. (1)

36 + 25 + 12g + 10f + c = 0

12g + 10f + c + 61 = 0 ……………………. (2)

Since centre lies on line 4x + y = 16, thus

-4g –f -16 = 0

4g + f + 16 = 0 ……………………………. (3)

Equation (2) – (1)

4g + 8f + 44 = 0

g + 2f + 11 = 0 ……………………………. (4)

8g + 2f + 32 = 0 …………………………... (5)

Equation (5) – (4)

7g + 21 = 0

g = -3
Putting the value of g in Equation (3)
-12 + f + 16 = 0
g = -3, f = -4
From Equation (1)
-24 – 8 + c + 17 = 0
-32 + 17 + c = 0
-15 + c = 0
c = 15
Thus the Equation of circle is: -
$x^2 + y^2 - 6x - 8y + 15 = 0$

Q.88 Find the value of $\lim\limits_{x\to b} \dfrac{e^{ax} - e^{ab}}{x - b}$ **(8)**

Ans:

$\lim\limits_{x\to b} \dfrac{e^{ax} - e^{ab}}{x - b}$ form 0/0

Using L-Hospital rule.

$\lim\limits_{x\to b} \dfrac{ae^{ax}}{1} = ae^{ab}$

Q.89 Differentiate y = tan x w.r.t. 'x' from first principle. **(6)**

Ans:

y = tanx

$y + \delta y = \tan(x + \delta x)$

$\dfrac{dy}{dx} = \underset{\delta x\to 0}{lt} \dfrac{\delta y}{\delta x}$

$= \lim\limits_{\delta x\to 0} \dfrac{\dfrac{\sin(x+\delta x)}{\cos(x+\delta)} - \dfrac{\sin x}{\cos x}}{\delta x}$

$= \lim\limits_{\delta x\to 0} \dfrac{\sin(x+\delta x)\cos x - \sin x\cos(x+\delta x)}{\cos x.\cos(x+\delta x)\delta x}$

$= \lim\limits_{\delta x\to 0} \dfrac{\sin(x+\delta x - x)}{\cos x.\cos(x+\delta x)\delta x}$

$= \lim\limits_{\delta x\to 0}\left(\dfrac{\sin \delta x}{\delta x}\right).\lim\limits_{\delta x\to 0} \dfrac{1}{\cos x}.\dfrac{1}{\cos(x+\delta x)}$

$= 1.\dfrac{1}{\cos^2 x} = \sec^2 x$

Q.90 Differentiate y = $x^{\sin x} + (\sin x)^x$ w.r.t 'x'. **(10)**

Ans:

$y = x^{\sin x} + (\sin x)^x$

Let $y_1 = x^{(\sin x)}$

$\log y_1 = \sin x \log x$

$\frac{1}{y_1}\frac{dy_1}{dx} = \cos x \log x + \frac{\sin x}{x}$

$\frac{dy_1}{dx} = x^{\sin x} \log x^{\cos x} + x^{\sin x}.\frac{\sin x}{x}$

Let $y_2 = (\sin x)^x$

$\log y_2 = x \log \sin x$

$\frac{1}{y_2}\frac{dy_2}{dx} = \log \sin x + \frac{x}{\sin x}\cos x$

$\therefore \frac{dy_2}{dx} = (\sin x)^x[\log \sin x + x \cot x]$

$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$

Q.91 Prove that straight line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y. **(8)**

Ans:

The point where the curve crosses the axis is given by put $x = 0 \Rightarrow y = b$. $Y = be^{\frac{-x}{a}}$

$\Rightarrow \frac{dy}{dx} = \frac{-b}{a}e^{\frac{-x}{a}} \Rightarrow \left(\frac{dy}{dx}\right)_{(0,b)} = -\frac{b}{a}$

Equation of tangent at the point (0, b)

$y - b = \frac{-b}{a}(x) \Rightarrow ay - ab = -bx$

$\Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$

Hence Proved.

Q.92 Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x-axis. **(8)**

Ans:

Required value = $2\int_0^a \pi y^2 dx$

$\Rightarrow 2\pi b^2\left(1 - \frac{x^2}{a^2}\right)_0^a$

$\Rightarrow 2\pi b^2\left[a - \frac{a}{3}\right] = \frac{4\pi ab^2}{3}$

Q.93 Prove that $\int_0^{\pi/2} \log \sin x dx = -\frac{\pi}{2} \log 2$. **(10)**

Ans:

$$I = \int_0^{\frac{\pi}{2}} \log \sin x$$

$$= \int_0^{\frac{\pi}{2}} \log \cos x$$

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \cos x dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \sin 2x - \int_0^{\frac{\pi}{2}} \log 2 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \frac{\pi}{2} \log 2$$

$$2x = t \Rightarrow 2dx = dt$$

$$2I = \int_0^{\pi} \log \sin t \frac{dt}{2} - \frac{\pi}{2} \log 2$$

$$2I = I - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = \frac{-\pi}{2} \log 2$$

Q.94 Solve $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$. **(6)**

Ans:

$$\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$$

Let $\sin^{-1} x = t \Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

$$\int t^3 dt = \frac{t^4}{4} + c = \frac{(\sin^{-1} x)^4}{4} + c$$

Q.95 Solve $3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$. **(8)**

Ans:

$3e^x \tan y dx + (1-e^x)\sec^2 y dy = 0$

$\frac{3e^x}{1-e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$

$I_1 + I_2 = 0$

$3\int\left(\frac{e^x}{1-e^x}\right)dx + \log\tan y = c$

Let $1-e^x = t$

$-e^x dx = dt$

$-3\int\frac{dt}{t} + \log\tan y = c$

$\Rightarrow \log\tan y - 3\log t = c$

$\Rightarrow \log\frac{\tan y}{t^3} = c$

$\Rightarrow \frac{\tan y}{(1-e^x)^3} = c$

$\tan y = c(1-e^x)^3$

Q.96 Solve $\left(1+x^2\right)\frac{dy}{dx} + 2xy - 4x^2 = 0$ subject to the initial condition y(0) = 0. **(8)**

Ans:

$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \left(\frac{4x^2}{1+x^2}\right)$

I.F = $\int\frac{2x}{1+x^2}dx = e^{\log(1+x^2)} = (1+x^2)$

$y.(1+x^2) = \int 4x^2 dx + c$

$y(1+x^2) = \frac{4x^3}{3} + c$

x = 0, y = 0

$\Rightarrow c = 0$

$\therefore \quad y = \frac{4x^3}{3(1+x^2)}$

Q.97 How many terms are there in a finite AP whose first and fifth terms are respectively –14 & 2 and the sum of terms is 40. **(8)**

Ans:

Let first term in AP be ‘a’ and ‘d’ be the common difference.
According to the given condition
First term a = -14
Fifth term a + 4d = 2
-14 + 4d = 2

4d = 16
d = 4
According to another condition

$$40 = \frac{n}{2}[2(-14)+(n-1)4]$$

$$= \frac{n}{2}[-28+4n-4]$$

$$= \frac{n}{2}(4n-32)$$

$$80 = -32n+4n^2$$

$$4n^2 - 32n - 80 = 0$$

$$n^2 - 8n - 20 = 0$$

$$n(n-10)+2(n-10) = 0$$

Neglecting n = -2 because no of terms cannot be negative
∴ The only possibility is n = 10.

Q.98 The sum of three numbers in G.P. is $13/12$ and their product is –1. Find the numbers. **(8)**

Ans:

Let the three numbers in G.P be $\frac{a}{r}$, a, ar
Then according to the first condition

$$\frac{a}{r}+a+ar = \frac{13}{12}$$

$$a\left(\frac{1}{r}+1+r\right) = \frac{13}{12}$$

$$a\left(\frac{1+r+r^2}{r}\right) = \frac{13}{12} \quad \text{.............................. (1)}$$

According to the second condition

$$\frac{a}{r}*a*ar = -1$$

$$a^3 = -1$$

$$a^3 = (-1)^3$$

a = -1 .... (2)

Substituting the value of *a* in equation (1)

$$-\left(\frac{1+r+r^2}{r}\right) = \frac{13}{12}$$

$$12+12r+12r^2 = -13r$$

$$12r^2+25r+12 = 0$$

$$r = \frac{-25 \pm \sqrt{625-576}}{24}$$

$$= \frac{-25 \pm \sqrt{49}}{24} = \frac{-25 \pm 7}{24}$$

r = $\frac{-18}{24}, \frac{-32}{24}$

r = $-\frac{3}{4}, -\frac{4}{3}$

a = -1, r = $-\frac{3}{4}$

Then the three numbers be

$\frac{a}{r}, a, ar$ ie

$\frac{-1}{-\frac{3}{4}}, -1, (-1)\left(\frac{3}{4}\right)$

$\frac{4}{3}, -1, \frac{3}{4}$

When a = -1, r = $-\frac{4}{3}$

Then the three numbers be

$\frac{a}{r}, a, ar$ ie

$\frac{-1}{-\frac{4}{3}}, -1, (-1)\left(-\frac{4}{3}\right)$

$\frac{3}{4}, -1, \frac{4}{3}$

Q.99 If $A + B + C = 180^\circ$, prove that

$$\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$ **(8)**

Ans:

Given A + B + C = 180

To Prove that $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

L.H.S.

$\cos A + \cos B - \cos C = (\cos A + \cos B) - \cos C$ ------------------------- (1)

$= \left(2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\right) - \cos C$

$= 2\cos\left(\frac{180-C}{2}\right)\cos\frac{A-B}{2} - \cos C$

$= 2\cos\left(90-\frac{C}{2}\right)\cos\frac{A-B}{2} - \cos C$ ------------- (2)

$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - \left(1 - 2\sin^2\frac{C}{2}\right)$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 1 + 2\sin^2\frac{C}{2}$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\frac{C}{2}\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\frac{180-(A+B)}{2}\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\left(90 - \frac{A+B}{2}\right)\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A}{2} - \frac{B}{2}\right) + \cos\left(\frac{A}{2} + \frac{B}{2}\right)\right]$$

$$= -1 + 2\sin\frac{C}{2}\left[2\cos\frac{A}{2} + \cos\frac{B}{2}\right]$$ **= R.H.S.**

Q.100 In any triangle ABC, prove that **(8)**

$$\frac{b^2 + c^2 - a^2}{a^2 + b^2 - c^2} = \frac{\tan C}{\tan A}$$

Ans:

L.H.S. $\frac{b^2 + c^2 - a^2}{a^2 + b^2 - c^2} = \dfrac{\frac{b^2 + c^2 - a^2}{2abc}}{\frac{a^2 + b^2 - c^2}{2abc}}$

$$= \dfrac{\frac{\cos A}{K\sin A}}{\frac{\cos C}{K\sin C}} = \frac{\tan C}{\tan A}$$ **= R.H.S.**

Q.101 Find the vertex, axis, focus, latus rectum and directrix of the parabola $x^2 + 2y - 3x + 5 = 0$. **(8)**

Ans:

The given equation is

$x^2 + 2y - 3x + 5 = 0$

$x^2 - 3x = -2y - 5$

$x^2 - 3x + \frac{9}{4} = -2y - 5 + \frac{9}{4}$

$\left(x - \frac{3}{2}\right)^2 = -2y - \frac{11}{4}$

$\left(x - \frac{3}{2}\right)^2 = -2\left(y + \frac{11}{8}\right)$ ------------------------------ (2)

$x-\frac{3}{2}=X,\ y+\frac{11}{8}=Y$

$X^2=-2Y$ -- (1)

Comparing it units $X^2=-4aY$

$4a=2,\quad a=\frac{1}{2}$

Vertex $\left(\frac{3}{2},-\frac{11}{8}\right)$

Axis $x-\frac{3}{2}=0$

Focus $\left(\frac{3}{2},-\frac{15}{8}\right)$

L.R. 2

Q.102 Find the equation of the circle which passes through the points (1, 1) & (2, 2) & whose radius is 1. **(8)**

Ans:

$x^2+y^2+2gx+2fy+c=0$ (1)

equation (1) passes through the point (1, 1)

$(1)^2+(1)^2+2g(1)+2f(1)+c=0$

2g + 2f + c = -2 (2)

equation (1) passes through the point (2, 2)

$(2)^2+(2)^2+2g(2)+2f(2)+c=0$

4g + 4f + c = -8 (3)

Also radius = 1

$g^2+f^2-c=1$ (4)

Solving equation (2) and (3)

-2g -2f = 6

g + f = -3 (5)

Solving equation (3) and (4)

4g + 4f + c = -8

$g^2+f^2-c=1$

$g^2+4g+f^2+4f=-7$ (6)

Solving equation (5) and (6)

$(-3-f)^2+4(-3-f)+f^2+4f=-7$

$9+f^2+6f-12-4f+f^2+4f=-7$

$2f^2+6f+4=0$

$f^2+3f+2=0$

$(f+1)(f+2)=0$

f = -1 f = -2

g = -3 + 1 = -2 g = -3 + 2 = -1

$g^2+f^2-c=1$ $g^2+f^2-c=1$

4 + 1 – c = 1   1 + 4 – c =1

c = 4   c = 4

Thus the required equation of the circle is

$x^2 + y^2 - 4x - 2y + 4 = 0$ & $x^2 + y^2 - 2x - 4y + 4 = 0$

Q.103 Find the equation of the straight line perpendicular to 7x + 9y – 3 = 0 and passing through (3, 8) **(8)**

Ans:

Equation of straight line perpendicular to 7x+ 9y + 3 = 0 is 9x – 7y + k = 0

It passes through (3, 8)

∵ Any line perpendicular to ax + by + c = 0 is given by bx + ay + k = 0

9(3) – 7(8) + k = 0

27 – 56 + k = 0

k = 29

Thus the required equation be

9x – 7y + 29 = 0

Q.104 Differentiate from the first principle the function y = sin 3x. **(8)**

Ans:

If f(x) = y = sin 3x

Using first principle

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(3x+3h) - \sin 3x}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{3x+3h+3x}{2}\right)\sin\left(\frac{3x+3h-3x}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{6x+3h}{2}\right)\sin\left(\frac{3h}{2}\right)}{h}$$

$$= \lim_{h\to 0} 2\cos\left(\frac{6x+3h}{2}\right)\frac{\sin\frac{3h}{2}}{\frac{3h}{2}}\times\frac{3}{2}$$

$$= 2\cos\left(\frac{6x}{2}\right)(1) * \frac{3}{2}$$

= 3 cos 3x.

Q.105 Evaluate $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$. **(8)**

Ans:

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} = \frac{\tan 0 - \sin 0}{\sin^3 0} = \frac{0-0}{0} = \frac{0}{0} \text{ Form}$$

$$\lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\sin x\left(\frac{1}{\cos x} - 1\right)}{\sin x . \sin^2 x}$$

$$\lim_{x \to 0} \frac{1-\cos x}{\cos x(1-\cos^2 x)} = \lim_{x \to 0} \frac{1-\cos x}{\cos x(1-\cos x)(1+\cos x)}$$

$$\lim_{x \to 0} \frac{1}{\cos x(1+\cos x)} = \frac{1}{2}$$

Q.106 Find the points of maxima or minima values of the function $y = x^3 - 18x^2 + 96x$. **(8)**

Ans:

$y = x^3 - 18x^2 + 96x$

Differentiating both sides w.r.t 'x'

$\frac{dy}{dx} = 3x^2 - 36x + 96$ --------------------------------- (1)

Put $\frac{dy}{dx} = 0$

$3x^2 - 36x + 96 = 0$

$x^2 - 12x + 32 = 0$

$x^2 - 8x - 4x + 32 = 0$

$(x-8)(x-4) = 0$

x = 8, 4

Differentiating (1) w.r.t x both side

$\frac{d^2y}{dx^2} = 6x - 36$

At x = 4, $\frac{d^2y}{dx^2} = 6(4) - 36 = -12 < 0$

$\therefore x = 4$ is a point of maxima and maximum value

$y = (4)^3 - 18(4)^2 + 96(4)$

= 64 – 18(16) + 384

= 64 -288 + 384 = 160

At x = 8, $\frac{d^2y}{dx^2} = 6 \times 8 - 36 = 12 > 0$

$\because x = 8$ is a point of minima and minimum value

$y = (8)^3 - 18(8)^2 + 96(8)$

= 512 – 1152 + 768

= 1280 – 1152

= 128

Q.107 Evaluate $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx$. **(8)**

Ans:

$$\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx$$

Put $a\cos^2 x + b\sin^2 x = t$

Differentiating both side w.r.t 'x'

$$a2\cos x(-\sin x) + b2\sin x(\cos x) = \frac{dt}{dx}$$

$$-(a\sin 2x) + (b\sin 2x) = \frac{dt}{dx}$$

$$(b-a)\sin 2x = \frac{dt}{dx}$$

$$\sin 2x dx = \frac{1}{b-a} dt$$

$$= \frac{1}{b-a}\int \frac{1}{t} dt$$

$$= \frac{1}{b-a}\log|t| + c \qquad \text{.....(1)}$$

$$\frac{1}{b-a}\log|a\cos^2 x + b\sin^2 x| + c \qquad \text{......(2)}$$

Q.108 Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$. **(8)**

Ans:

Put $x = \tan\theta,\ dx = \sec^2\theta d\theta$

x = 0, θ = 0

x = 1, $\theta = \frac{\pi}{4}$ (1)

Let $I = \int_0^{\frac{\pi}{4}} \log(\tan x + 1) dx$

$$\int_0^{\frac{\pi}{4}} \log(\tan x + 1) dx \qquad (1)$$

Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} [\log 2 - \log(1+\tan x)]dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$$

$$= \log 2\int_0^{\frac{\pi}{4}} 1dx - 1$$

$$2I = \log 2.[x]_0^{\frac{\pi}{4}}$$

$$2I = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2$$

Q.109 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(8)**

Ans:

The equation of the curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

The curve is symmetrical about the axis

$\therefore$ Area enclosed by the ellipses

= 4 (area enclosed by the ellipse and coordinate axes in first quadrant)

Required area = $4\int_0^a ydx$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}.\frac{1}{2}\left(x\sqrt{a^2 - x^2}\right)_0^a + \frac{4b}{a}.\frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right)\right)_0^a$$

$$= 2ab\frac{\pi}{2}$$

$= \pi ab$ sq units

Q.110 Solve $x^2 dy + y(x + y)dx = 0$. **(8)**

Ans:

$$x^2 dy + (xy + y^2)dx = 0$$

$$x^2 dy = -(xy + y^2)dx$$

$$\frac{dy}{dx} = -\frac{xy + y^2}{x^2}$$

Let y = vx (homogenous form)

Differentiating both side w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\frac{xvx + v^2x^2}{x^2}$$

$$v + x\frac{dv}{dx} = -(v + v^2)$$

$$x\frac{dv}{dx} = -2v - v^2$$

$$\frac{1}{v^2 + 2v}dv = -\frac{1}{x}dx$$

$$\frac{1}{v^2 + 2v}dv + \frac{1}{x}dx = 0$$

Integrating both side

$$\int \frac{1}{v^2 + 2v}dv + \int \frac{1}{x}dx = \int 0dx$$

$$\int \frac{1}{v^2 + 2v + 1 - 1}dv + \int \frac{1}{x}dx = \int 0dx$$

$$\frac{1}{2(1)}\log\left|\frac{v + 1 - 1}{v + 1 + 1}\right| + \log|x| = 2\log|c| \qquad (2)$$

$$\log\left|\frac{\frac{y}{x}}{\frac{y}{x}+2}\right| + 2\log|x| = 2\log|c|$$

$$\log\left|\frac{y}{y+2x}\right| + \log\left|x^2\right| = 2\log\left|c^2\right|$$

$$\log\left|\frac{yx^2}{y+2x}\right| = \log|c_1|$$

Taking antilog on both sides

$$\frac{yx^2}{2x+y} = c_1$$

$$yx^2 = c_1(2x+y)$$

Q.111 Solve $\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3$. **(8)**

Ans:

Comparing the above equation with $\frac{dy}{dx} + py = Q$

$$P = \frac{1}{x},\ Q = x^3 - 3$$

I.F = $e^{\int Pdx} = e^{\int \frac{1}{x}dx}$ (1)

I.F = $e^{\log x} = x$ (2)

Required solution

$$y(I.F) = \int Q.(I.F)dx + c$$

$$y.x = \int (x^3 - 3)xdx + c$$

$$xy = \int (x^4 - 3x)dx + c$$

$$xy = \frac{x^5}{5} - \frac{3x^2}{2} + c \qquad (3)$$

$$y = \frac{x^4}{5} - \frac{3x}{2} + \frac{c}{x}$$

Q.112 If $(x + i\,y)^{1/3} = a + i\,b$ where $x, y, a, b \in R$ Show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$. **(7)**

Ans:

We have $(x+iy)^{1/3} = a+ib$

$$x + iy = (a+ib)^3 = (a^3 - 3ab^2) + i(3a^2b - b^3) \Rightarrow x = a^3 - 3ab^2,\ y = 3a^2b - b^3$$

$$\frac{x}{a} = a^2 - 3b^2,\ \frac{y}{b} = 3a^2 - b^2$$

$$\frac{x}{a}+\frac{y}{b}=a^2-3b^2+3a^2-b^2 \qquad =4a^2-4b^2 \qquad =4\left(a^2-b^2\right)$$

Q.113 Put the following in the form $r\left(\cos\theta + i\sin\theta\right)$, where r is a positive real number and $-\pi < \theta \le \pi$. **(7)**

Ans:

$\left(1+7\,i\right)/\left(2-i\right)^2$

Let $r\ (\cos\theta + i\sin\theta) = \dfrac{1+7i}{(2-i)^2}$

$$=\frac{1+7i}{4-1-4i} \quad =\frac{1+7i}{3-4i} \quad =\frac{(1+7i)(3+4i)}{9+16} \quad =\frac{-25+25i}{25} \quad =-1+i$$

$$\Rightarrow r\cos\theta=-1, r\sin\theta=1\ ,\quad r^2=2 \Rightarrow r=\sqrt{2}\quad ,\sin\theta=\frac{1}{\sqrt{2}} \quad \Rightarrow \theta=\frac{3\pi}{4}$$

$$\frac{1+7i}{(2-i)^2}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

Q.114 A two-digit number is four times the sum and three times the product of the digits. Find the number. **(7)**

Ans:

Let the number is $10x+y$ where x is tens digit and y is unit digit.

Given $10x+y=4(x+y)$ (1)

and $10x+y=3xy$ (2)

From (1),we get

$6x=3y \Rightarrow y=2x$

Using this in (2), 10x + 2x = 3x(2x) or 12x = 6x^2 or x^2 -2x = 0 ,
$\Rightarrow$ x = 0, x = 2.

If x = 0, then y = 0 which is inadmissible. If x = 2 then y = 4, hence the required number is 10(2) +4=24

Q.115 Solve the simultaneous equations: $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2};\ x+y=10$. **(7)**

Ans:

We have

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2} \quad \text{.....................(1)}$$

x + y = 10 ....(2)

$$(1) \Rightarrow \frac{x+y}{\sqrt{xy}}=\frac{5}{2} \quad \Rightarrow \frac{10}{\sqrt{xy}}=\frac{5}{2}, \text{ using (2).}$$

$\Rightarrow xy=16$

Thus, the given system of equations is

x + y = 10, xy = 16 $\Rightarrow$ y = 10 – x and x (10- x) = 16

$\Rightarrow x^2 - 10x + 16 = 0 \Rightarrow x = 2, 8$

If x = 2, y = 8. And if x= 8, y = 2.

Hence roots are x = 2, y = 8 and x = 8. y = 2

Q.116 The diagonal of a square lies along the line $8x-15y=0$ and one vertex of the square is (1, 2). Find the equations of the sides of the square. **(7)**

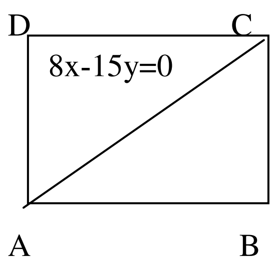
Ans:

Let ABCD be a square such that the diagonal AC is 8x – 15y = 0 and the vertex B is (1,2). We have to find the sides passing through B clearly, sides BA and BC pass through B(1,2) and are inclined at an angle of 45° to the diagonal AC. So, the equations of BA and BC are

$y-2=\dfrac{m\pm\tan 45^0}{1\mp m\tan 45^0}(x-1)$ where m is the slope of the line

$8x-15y=0$ i,e $m=\dfrac{8}{15}$ $\Rightarrow y-2=\dfrac{8/15\pm 1}{1\mp 8/15}(x-1)$ or $y-2=\dfrac{8/15+1}{1-8/15}(x-1)$

and $y-2=\dfrac{8/15-1}{1+8/15}(x-1) \Rightarrow 23x-7y-9=0$ *and* $7x+23y-53=0$ ....(3)



Coordinates of A, C are $\left(\dfrac{135}{289},\dfrac{72}{289}\right),\left(\dfrac{795}{289},\dfrac{424}{289}\right)$

other two sides are parallel to the sides (3)

hence are $23x-7y=c_1$, $7x+23y=c_2$

These respectively pass through C and A. We can find c_1, c_2 by using this condition.

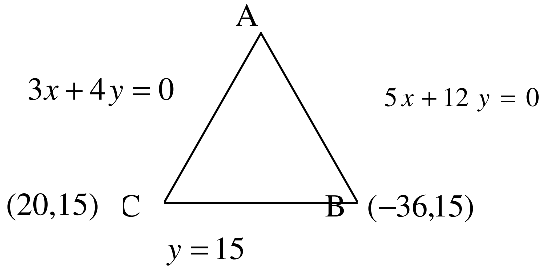
Q.117 Find the centroid and incentre of the triangle whose sides have the equations $3x-4y=0,\ 12y+5x=0$ and $y-15=0$. **(7)**

Ans:

Let ABC be the triangle whose sides BC,CA and AB have the equations

y -15 = 0, 3x - 4y = 0, 5x +12y = 0 respectively. Solving these equations pair wise we can obtain the coordinates of the vertices A,B,C as A(0,0), B(-36,15), C(20,15) respectively

A (0,0)



Centroid:

The coordinates of centroid are

$$\left(\frac{0-36+20}{3},\frac{0+15+15}{3}\right)=\left(\frac{-16}{3},10\right)$$

For Incentre:

We have

a =BC= $\sqrt{(-36-20)^2+(15-15)^2}=56$

b=CA= $\sqrt{20^2+15^2}=25$

c=AB= $\sqrt{(-36-0)^2+(15-0)^2}=39$

$\therefore$ Coordinates of incentre are

$$\left(\frac{56\times 0+25\times -36+39\times 20}{56+25+39},\ \frac{56\times 0+25\times 15+39\times 15}{56+25+39}\right)=(-1,8)$$

Q.118 (i) Find the equation of the circle which touches both the axes and whose radius is 5.

(ii) Find the coordinates of the centre and radius of the circle

$$2x^2+2y^2-3x+5y=7. \qquad \textbf{(7)}$$

Ans:

(i) The equation of circles which touch both the axes are

$(x\pm a)^2+(y\pm a)^2=a^2$

and $(x\pm a)^2+(y\mp a)^2=a^2$

Here $h=k=\pm a$ and radius equals 5, Therefore circles are

$(x\pm 5)^2+(y\pm 5)^2=25$ and $(x\pm 5)^2+(y\mp 5)^2=25$

$x^2+y^2\pm 10x\pm 10y+25=0$ and $x^2+y^2\pm 10x\mp 10y+25=0$

(ii) In the given equation the coefficients of x^2 and y^2 one not unity.

We have to re-write the equation to make the coefficients of x^2 and y^2 unity. We have $2x^2+2y^2-3x+5y=7$

$\Rightarrow x^2+y^2-\frac{3}{2}x+\frac{5}{2}y=\frac{7}{2}$

The coordinates of centre are $\left(\frac{3}{4},\frac{-5}{4}\right)$ and radius= $\sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{-5}{4}\right)^2+\frac{7}{2}}$

$=\sqrt{\frac{9}{16}+\frac{25}{16}+\frac{7}{2}} \quad =\frac{3}{4}\sqrt{10}$

Q.119 Find the equation of a circle passing through the points (1, 2) and (3, 0) and cutting an intercept 4 on the x-axis. **(7)**

Ans:

Let the equation of the circle be

$x^2+y^2+2gx+2fy+c=0$ (1)

Since it passes though the points (1,2)and (3,0)

$1 + 4 + 2g + 4f + c = 0$

$\Rightarrow 2g + 4f + c = -5$ (2)

and $9+6g+c=0 \Rightarrow 6g+c=-9$ (3)

Also the length of x-intercept is 4

$\Rightarrow 2\sqrt{g^2 - c} = 4$

$\Rightarrow g^2-c=4$ (4)

From (3) and(4)

$g^2-(-9-6g)=4$

$g^2+6g+5=0$

$(g+5)(g+1)=0 \quad g = -1, -5$

From (3),if g = -1, c = -3

if g = -5, c = 21

Also from (2) if g = -1, c = 3 then f = 0

and If g = 5, c = 21 then f = -4

Equations are $x^2+y^2-2x-3=0, x^2+y^2-10x-8y+21=0$

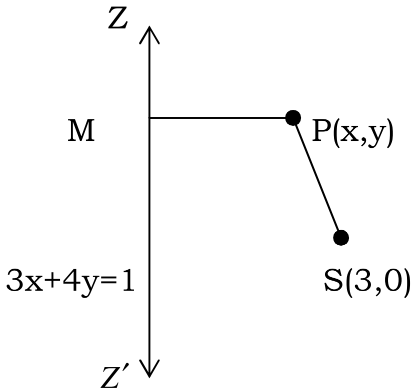
Q.120 Find the equation of the parabola whose focus is (3, 0) and the directrix is 3x +4y = 1. **(7)**

Ans:

Let P (x, y) be any point on the parabola whose focus is S$(3,0)$ and the directrix $3x+4y=1$

Draw PM perpendicular to $3x+4y=1$. Then, by definition for parabola

$SP=PM \Rightarrow SP^2 = PM^2$



$$\Rightarrow (x-3)^2 + y^2 = \left(\frac{3x+4y-1}{\sqrt{3^2+4^2}}\right)^2$$

or $x^2 - 6x + 9 + y^2 = \dfrac{(3x+4y-1)^2}{25}$

or $16x^2 + 9y^2 - 24xy - 144x + 8y + 224 = 0$ is the required equation of parabola.

Q.121 Find the equation of an ellipse whose foci are at (± 3, 0) and which passes through (4, 1). **(7)**

Ans:
Let the equation of ellipse be

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The coordinates of foci are

(± a e,o) $\Rightarrow ae = 3 \Rightarrow e = \dfrac{3}{a}$. But $b^2 = a^2(1-e^2) \Rightarrow b^2 = a^2\left[1 - \dfrac{9}{a^2}\right]$.............(1)

Also the ellipse passes though (4,1)

$\Rightarrow \dfrac{16}{a^2} + \dfrac{1}{b^2} = 1 \Rightarrow \dfrac{1}{b^2} = 1 - \dfrac{16}{a^2} = \dfrac{a^2-16}{a^2}$ or $b^2 = \dfrac{a^2}{a^2-16}$. Substituting in (1)

$$\Rightarrow a^2\left(1 - \frac{9}{a^2}\right) = \frac{a^2}{a^2-16}$$

or $a^2 - 9 = \dfrac{a^2}{a^2-16}$

or $(a^2-9)(a^2-16) - a^2 = 0$

or $a^4 - 26a^2 + 144 = 0$

or $(a^2-18)(a^2-8) = 0$

$a^2 = 18, a^2 = 8$

If $a^2 = 18, b^2 = \dfrac{18}{18-16} = 9$

If $a^2 = 8, b^2 = \dfrac{18}{8-16} = -1$ (not possible) $\therefore a^2 = 18,\ b^2 = 9$

Equation of ellipse is $\dfrac{x^2}{18} + \dfrac{y^2}{9} = 1$

Q.122 If $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$, prove that $\dfrac{dy}{dx} = -\sqrt{\dfrac{1-y^2}{1-x^2}}$. **(7)**

Ans:
Given

$y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$

Diff. w.r to x

$$\frac{dy}{dx}\sqrt{1-x^2}+y\frac{(-2x)}{2\sqrt{1-x^2}}+\sqrt{1-y^2}+x\frac{(-2y)}{2\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}\left[\sqrt{1-x^2}-\frac{xy}{\sqrt{1-y^2}}\right]+\sqrt{1-y^2}-\frac{xy}{\sqrt{1-x^2}}=0$$

or $\frac{dy}{dx}\left[\frac{\sqrt{1-x^2}\sqrt{1-y^2}-xy}{\sqrt{1-y^2}}\right]=\frac{xy-\sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-x^2}}$

or $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$

Q.123 (i) A man 2 metres high walks at a uniform speed of 6 metres per minute away from a lamp post, 5 metres high. Find the rate at which the length of his shadow increases.

(ii) Use differentials to find the approximate value of $\sqrt{0.037}$. **(7)**

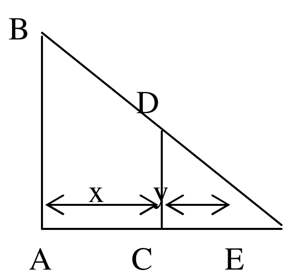
Ans:

(i) Let AB be the lamp-post. Let at any time t, the man CD be at a distance x meters from the lamp-post and y meters be the length of his shadow CE.

Then $\frac{dx}{dt}$ = 6 meters / minute (given)

Now, triangle ABE and CDE are similar, therefore

$$\frac{AB}{CD}=\frac{AE}{CE} \Rightarrow \frac{5}{2}=\frac{x+y}{y} \Rightarrow 3y=2x$$



$$\Rightarrow 3\frac{dy}{dt}=2\frac{dx}{dt}\Rightarrow 3\frac{dy}{dt}=12\Rightarrow \frac{dy}{dt}=4$$

Thus, the shadow increases at the rate of 4 meters/minute.

(ii) Let y = f (x) = $\sqrt{x}$

x = 0.040 and x + Δx = 0.037

then Δx = -0.003.

For x = .040, y = .2

Let dx = Δx = -0.003

Now, $y=\sqrt{x}\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{x}}$

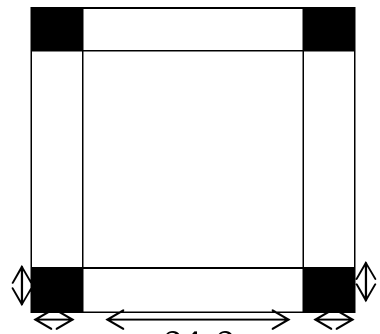
$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0.040}=\frac{1}{0.4}$. By using $dy=\frac{dy}{dx}dx$ we get $dy=\frac{1}{.4}(-0.003)=\frac{-3}{400}$

Now, Δy is the approximately equal to dy, so $\Delta y = \frac{-3}{400}$.

Hence $\sqrt{0.037} = 0.2 + \Delta y = 0.1925$

Q.124 A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum. **(7)**

Ans:



Let x cm be the length of a side of the square which is cut-off from each corner of the plate. Then sides of the box as shown in fig. above are 24 - 2x, 24 - 2x and x.
Let V be the volume of the box .Then

$$V = (24-2x)^2 \cdot x$$
$$= 4x^3 - 96x^2 + 576x$$
$$\frac{dV}{dx} = 12x^2 - 192x + 576$$
$$\frac{d^2V}{dx^2} = 24x - 192$$

For maximum or minimum V,

$$\frac{dV}{dx} = 0 \Rightarrow 12x^2 - 192x + 576 = 0 \Rightarrow x = 4,12$$

But x = 12 is not possible , thus x = 4

Now, $\left(\frac{d^2V}{dx^2}\right)_{x=4} = 24 \times 4 - 192 = 96 - 192$

$= -96 < 0$

Thus, V is maximum
when x = 4
Hence, the volume of the box is maximum when the side of the square cut off is 4 cm.

Q.125 Evaluate the following integrals

(i) $\int \sqrt{\frac{1-x}{1+x}}\, dx$ (ii) $\int \frac{1}{1+\sin x}\, dx$. **(7)**

Ans:

(i) $\int\sqrt{\frac{1-x}{1+x}}dx = \int\sqrt{\frac{(1-x)^2}{1-x^2}}dx = \int\frac{1-x}{\sqrt{1-x^2}}dx = \int\frac{dx}{\sqrt{1-x^2}} - \int\frac{x}{\sqrt{1-x^2}}dx$

$= \sin^{-1}x - I + C$ --------------------(1)

$I = \int\frac{x}{\sqrt{1-x^2}}dx$

Let $1-x^2 = z^2 \Rightarrow -2xdx = 2zdz \Rightarrow$

$I = \int\frac{-zdz}{z} = -\int dz = -z = -\sqrt{1-x^2}$

From (1)

$\int\sqrt{\frac{1-x}{1+x}}dx = \sin^{-1}x + \sqrt{1-x^2} + C$

(ii) $\int\frac{1}{1+\sin x}dx = \int\frac{1-\sin x}{1-\sin^2 x}dx$

$= \int\frac{1-\sin x}{\cos^2 x}dx = \int\sec^2 x dx - \int\sec x\tan x dx$ = tan x – sec x +c

Q.126 Draw the rough sketch of area enclosed by curves $y^2+1 = x$, and $x = 2$. Also find this area. **(7)**

Ans:

The point of intersections of

$y^2 = x-1$, and $x = 2$ are (2,1) and (2,-1).

Required area is shaded area in the figure

$= \left|\int_{-1}^{1}(x_2 - x_1)dy\right| = \left|\int_{-1}^{1}(y^2+1-2)dy\right| = \left|\int_{-1}^{1}(y^2-1)dy\right| = \left|\left(\frac{y^3}{3}-y\right)_{-1}^{1}\right|$

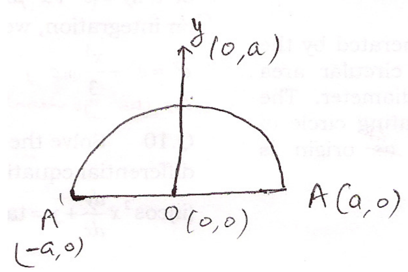
$= \left|\frac{1}{3}-1+\frac{1}{3}-1\right| = \left|\frac{2}{3}-2\right| = \frac{4}{3}$. Area$=\frac{4}{3}$ Sq units

Q.127 Using integration, show that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. **(7)**

Ans:

The sphere is generated by the revolution of a semi circular area about its bounding diameter.

The equation of the generating circle of radius 'a' with centre at origin is $x^2 + y^2 = a^2$



Let A A' be the bounding diameter about which the semi-circle revolves

$\therefore$ The required volume of the sphere

$$= 2\int_0^a \pi y^2 dx$$

$$= 2\pi\int_0^a \left(a^2 - x^2\right)dx \quad = 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a \quad = 2\pi\left[a^3 - \frac{a^3}{3}\right] \quad = \frac{4}{3}\pi a^3$$

Q.128 Solve the following differential equations

(i) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$. (ii) $\cos^2 x \frac{dy}{dx} + y = \tan x$

(iii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$. **(14)**

Ans:

(i) $\frac{dy}{dx} = e^x . e^{-y} + x^2 e^{-y}$

Separating the variables

$\frac{dy}{e^{-y}} = \left(e^x + x^2\right)dx$ or $e^y dy = \left(e^x + x^2\right)dx$,

On integration, we have

$e^y = e^x + \frac{x^3}{3} + c$, c arbitrary, as the general solution.

(ii) $\cos^2 x \frac{dy}{dx} + y = \tan x \Rightarrow \frac{dy}{dx} + \sec^2 x\, y = \sec^2 x \tan x$

This is linear differential equation

I.F = $e^{\int \sec^2 x dx} = e^{\tan x}$

Solution is

$y \cdot e^{\tan x} = \int e^{\tan x} . \sec^2 x \tan x dx$ +C

Let tan x = t, then $\sec^2 x \; dx = dt$ and integral on r.h.s. becomes

$\int e^t \cdot t dt = te^t - \int 1 \cdot e^t dt \quad = te^t - e^t$ s $= \tan x\, e^{\tan x} - e^{\tan x}$

$\therefore$ Solution is $y\, e^{\tan x} = e^{\tan x}(\tan x - 1) + C$

or $y = (\tan x - 1) + Ce^{-\tan x}$

(iii) $\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$ ------------(1)

Let $y = e^{mx}$ is the solution of (1), then auxiliary equation is

$m^2 - 5m + 6 = 0 \Rightarrow$ m = 2, 3

$C \cdot F \cdot = c_1 e^{2x} + c_2 e^{3x}$

$$P \cdot I = \frac{1}{D^2 - 5D + 6} e^{4x} = \frac{1}{16 - 20 + 6} e^{4x} = \frac{1}{2} e^{4x}$$

The general solution of differential equation is

y = CF + PI $= c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x}$. Where c_1, c_2 arbitrary

Q.129 (i) Find a unit vector perpendicular to both the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $\hat{i}+2\hat{j}-\hat{k}$.

(ii) If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

$$\sin\frac{\theta}{2}=\frac{1}{2}\left|\hat{a}-\hat{b}\right|$$

(iii) Find the moment of the couple formed by the forces $5\hat{i}+\hat{k}$ and $-5\hat{i}-\hat{k}$ acting at the points $(9,-1,2)$ and $(3,-2,1)$ respectively. **(14)**

Ans:

(i) Let the unit vector perpendicular to both the vectors is $\overline{C}=ai+bj+ck$

Let $\overline{A}=i-2j+3k$, $\overline{B}=i+2j-k$

$\overline{A}$ and $\overline{C}$ are perpendicular to each other

$\Rightarrow \overline{A}\cdot\overline{C}=0 \Rightarrow (i-2j+3k)\cdot(ai+bj+ck)=0$

$\Rightarrow a-2b+3c=0$ ------------------(1)

Also $\overline{B}$ and $\overline{C}$ are perpendicular

$\Rightarrow \overline{B}\cdot\overline{C}=0 \Rightarrow (i+2j-k)\cdot(ai+bj+ck)=0$

$\Rightarrow a+2b-c=0$ -------------------(2)

from (1) and (2)

$$\frac{a}{2-6}=\frac{b}{3+1}=\frac{c}{2+2}$$

$$\frac{a}{-4}=\frac{b}{4}=\frac{c}{4}=\lambda(say)$$

$\Rightarrow a=-4\lambda,\ \ \mathrm{b}=4\lambda,\ \ \mathrm{c}=4\lambda \ \Rightarrow \overline{C}=(-4\lambda)i+4\lambda j+4\lambda k$

$\Rightarrow \left|\overline{C}\right| = \sqrt{(-4\lambda)^2+(4\lambda)^2+(4\lambda)^2} = \sqrt{16\lambda^2+16\lambda^2+16\lambda^2} = 4\sqrt{3}\lambda$

unit normal vector

$$=\frac{C}{\left|\overline{C}\right|}=\frac{-4\lambda i+4\lambda j+4\lambda k}{4\sqrt{3}\lambda} = \frac{-i+j+k}{\sqrt{3}}$$

(ii) $\hat{a}\cdot\hat{b}=1\cdot1\cdot\cos\theta=\cos\theta$

Now $\left|\hat{a}-\hat{b}\right|^2=\left(\hat{a}-\hat{b}\right)^2 = (\hat{a})^2+\left(\hat{b}\right)^2-2\hat{a}\cdot\hat{b} = 1+1-2\cos\theta$

$$=2-2\left(1-2\sin^2\frac{\theta}{2}\right) = 4\sin^2\frac{\theta}{2}$$

$$\therefore 4\sin^2\frac{\theta}{2}=\left|\hat{a}-\hat{b}\right|^2 \quad \Rightarrow 2\sin\frac{\theta}{2}=\left|\hat{a}-\hat{b}\right| \quad \Rightarrow \sin\frac{\theta}{2}=\frac{1}{2}\left|\hat{a}-\hat{b}\right|$$

(iii) $M=r_1\times f_1+r_2\times f_2$

Here $r_1=9i-j+2k,\ r_2=3i-2j+k$

$f_1=5i+k,\quad f_2=-5i-k$

Now

$$r_1 \times f_1 = \begin{vmatrix} i & j & k \\ 9 & -1 & 2 \\ 5 & 0 & 1 \end{vmatrix} \quad = -i + j + 5k$$

$$r_2 \times f_2 = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ -5 & 0 & -1 \end{vmatrix} \quad = 2i - 2j - 10k$$

$M = i - j - 5k.$

Q.130 Find the term independent of x in the expansion of

$$\left\{3x - \frac{2}{x^2}\right\}^{15}$$ **(7)**

Ans:

Given $\left[3x - \frac{2}{x^2}\right]^{15}$. Let (r+1)th term be independent of x.

Now $T_{r+1} = 15_{C_r} (3x)^{15-r}\left(-\frac{2}{x^2}\right)^r \quad = 15_{C_r} 3^{15-r}(-2)^r x^{15-3r}$

For this term be independent of x, we must have

15-3r = 0 $\Rightarrow r = 5$, So, 6th term is independent of x.

$T_6 = 15_{C_5} 3^{10}(-2)^5 \quad = -15_{C_5} 3^{10} 2^5$

Q.131 If $A + B + C = \Pi$ prove that

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$ **(7)**

Ans:

Given $A + B + C = \pi$

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} \quad = \sin^2\frac{A}{2} + \sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$= \sin^2\frac{A}{2} + \sin\left(\frac{\pi - A}{2}\right)\cos\left(\frac{B-C}{2}\right) \quad = \sin^2\frac{A}{2} + \cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right)$$

$$= 1 - \cos^2\frac{A}{2} + \cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right) \quad = 1 - \left[\cos\frac{A}{2}\left(\cos\frac{A}{2} - \sin\left(\frac{B-C}{2}\right)\right)\right]$$

$$= 1 - \cos\frac{A}{2}\left[\cos\left(\frac{\pi}{2} - \frac{B+C}{2}\right) - \sin\left(\frac{B-C}{2}\right)\right] = 1 - \cos\frac{A}{2}\left[\sin\left(\frac{B+C}{2}\right) - \sin\left(\frac{B-C}{2}\right)\right]$$

$$= 1 - \cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2} \quad - \sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= 1 - \cos\frac{A}{2}\cdot 2\cos\frac{B}{2}\sin\frac{C}{2} \quad = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Q.132 If $x = a + b, y = a\alpha + b\beta$ $z = a\beta + b\alpha$ where α, β are complex cube roots of unity show that $xyz = a^3 + b^3$. **(7)**

Ans:
Given $x = a + b, \quad y = a\alpha + b\beta$
and $z = a\beta + b\alpha$
Let $\alpha = w, \beta = w^2$
$x = a + b, y = aw + bw^2$
$z = aw^2 + bw$
Now $xyz = (a+b)(aw + bw^2)(aw^2 + bw)$
$= (a+b)\left[a^2w^3 + b^2w^3 + abw^2 + abw^4\right]$
$= (a+b)\left[a^2 + b^2 + ab(w^2 + w)\right]$
$= (a+b)(a^2 + b^2 - ab)$
$= a^3 + b^3$

Q.133 If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ be equal prove that either $a = 0$ or $a^3 + b^3 + c^3 = 3abc$. **(7)**

Ans:
Given that the roots of
$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal.
$\Rightarrow$ The discriminant of the equation is zero
$4(a^2 - bc)^2 - 4(c^2 - ab)(b^2 - ac) = 0$
or $(a^4 + b^2c^2 - 2a^2bc) - (b^2c^2 - ab^3 - ac^3 + a^2bc) = 0$ $a^4 + ab^3 + ac^3 - 3a^2bc = 0$

$a\left[a^3 + b^3 + c^3 - 3abc\right] = 0 \Rightarrow$ *either* $a = 0$ or $a^3 + b^3 + c^3 = 3abc$

Q.134 Find the derivative of $\sin x^2$ from the first principles. **(7)**

Ans:
let $f(x) = \sin x^2$. Then

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

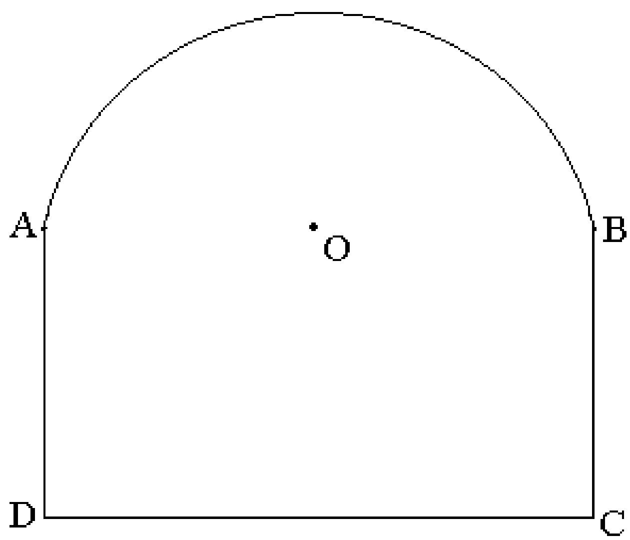
$$= \lim_{h \to 0} \frac{\sin(x+h)^2 - \sin x^2}{h}$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{2hx + h^2}{2}\right)\cos\left(\frac{2x^2 + 2hx + h^2}{2}\right)}{h}$$

$$\lim_{h\to 0}\frac{2\sin\left(\frac{2hx+h^2}{2}\right)}{h\left(\frac{2x+h}{2}\right)}\left(\frac{2x+h}{2}\right)\cdot\cos\left(\frac{2x^2+2hx+h^2}{2}\right)=\lim_{h\to 0}\frac{\sin\left(\frac{2hx+h^2}{2}\right)}{h\frac{2x+h^2}{2}}\cdot\lim_{h\to 0}(2x+h)$$

$$\cdot\lim_{h\to 0}\cos\left(\frac{2x^2+2hx+h^2}{2}\right)=1\cdot 2x\cdot\cos x^2=2x\cos x^2$$

Q.135 Take A semicircle with a rectangle on its diameter as shown in the figure below. If the perimeter of the figure is 20 feet, find its dimension in order that its area may be maximum.



(7)

Ans:

Let ABCD consists of a rectangle and let the semi-circle be described on side AB as diameter. Let AB=2x and AC = 2y. Let P be the perimeter and A be the area of fig. then

$P = 2x + 4y + \pi x$ ------------------(1)

$A = (2x)(2y) + \frac{\pi x^2}{2}$ ------------------(2)

$A = 4xy + \frac{\pi x^2}{2} = x[20 - 2x - \pi x] + \frac{\pi x^2}{2}$ $[\text{Given } P = 20]$ $= 20x - 2x^2 - \pi x^2 + \frac{\pi x^2}{2}$

$= 20x - 2x^2 - \frac{\pi x^2}{2}$, $\frac{dA}{dx} = 20 - 4x - \pi x$, $\frac{d^2A}{dx^2} = -4 - \pi$

For maxima or minima, $\frac{dA}{dx} = 0$, Thus $20 - (4+\pi)x = 0 \Rightarrow x = \frac{20}{4+\pi}$

Also $\frac{d^2A}{dx^2} = -4 - \pi < 0$ for all values of x. Thus, A is maximum when

$x = \frac{20}{4+\pi}$. From(1), $20 = 2\left(\frac{20}{4+\pi}\right) + 4y + \pi\left(\frac{20}{4+\pi}\right)$

$$= (2+\pi)\left(\frac{20}{4+\pi}\right) + 4y \Rightarrow 4y = 20 - \frac{20(2+\pi)}{4+\pi}$$

$$= \frac{20(4+\pi) - 20(2+\pi)}{4+\pi} = \frac{80 + 20\pi - 40 - 20\pi}{4+\pi}$$

$4y = \frac{40}{4+\pi}$ $y = \frac{10}{4+\pi}$. So, dimensions of rectangle are

$2x = \frac{40}{4+\pi}$, $2y = \frac{20}{4+\pi}$ and semicircle top has radius $\frac{20}{4+\pi}$

Q.136 Evaluate $\underset{x\to\infty}{\text{Lt}} \left(\sqrt{(x^2+x+1)} - x\right)$. **(6)**

Ans:

$$\lim_{x\to\infty}\left[\sqrt{x^2+x+1} - x\right] = \lim_{x\to\infty}\left[\frac{\left(\sqrt{x^2+x+1}\right)^2 - x^2}{\sqrt{x^2+x+1} + x}\right] = \lim_{x\to\infty}\left[\frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x+1}{\sqrt{x^2+x+1}+x}\right]$$

Divide by x $= \lim_{x\to\infty}\left[\frac{\frac{1}{x}+1}{\sqrt{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}+1}\right] = \frac{1}{\sqrt{1+0+0}+1} = \frac{1}{1+1} = \frac{1}{2}$ Ans.

Q.137 The rectangular co-ordinates of a point on the curve are $x = 3\cos\theta - \cos^3\theta,\ y = 3\sin\theta - \sin^3\theta$. Find the equation of the normal at any point on the curve and show that at the point with $\theta = \pi/4$, the normal passes through the origin. **(8)**

Ans:

Here $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 3\sin^2\theta\cos\theta}{-3\sin\theta + 3\cos^2\theta\sin\theta} = \frac{3\cos\theta(1-\sin^2\theta)}{-3\sin\theta(1-\cos^2\theta)}$$

$= \frac{-\cos^3\theta}{\sin^3\theta}$ Equation of normal is

$$y - y_1 = \frac{-dx}{dy}(x - x_1),\ y - 3\sin\theta + \sin^3\theta = \frac{\sin^3\theta}{\cos^3\theta}\left(x - 3\cos\theta + \cos^3\theta\right)$$

At $\theta = \frac{\pi}{4}$, $y - \frac{3}{\sqrt{2}} + \frac{1}{2\sqrt{2}} = x - \frac{3}{\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$y = x \Rightarrow$ The equation of normal passes through origin,

Q.138 Show that the curves $y^2 = 4ax$ and $ay^2 = 4x^3$ intersect each other at point (a, 2 a) at an angle $\tan^{-1}\frac{1}{2}$. **(7)**

Ans:
Solving for (x,y)

$y^2 = 4ax, \quad ay^2 = 4x^3$

$a(4ax) = 4x^3, 4a^2x = 4x^3, x = a \; y = 2a$ and $x = 0, y = 0$

A point of intersection is (a,2a)

Now, $\theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$

Here slopes at point $(a,2a)$ are $m_1 = \frac{4a}{2y} = 1$ and $m_2 = \frac{12x^2}{2ay} = \frac{12a^2}{4a^2} = 3$

$\theta = \tan^{-1}\left(\frac{3-1}{1+3}\right) = \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Q.139 Differentiate $\sin^{-1}\frac{2x}{1+x^2}$ with respect to $\cos^{-1}\frac{1-x^2}{1+x^2}$. **(7)**

Ans:

Let $u = \sin^{-1}\frac{2x}{1+x^2}$, $v = \cos^{-1}\frac{1-x^2}{1+x^2}$

Let $x = \tan\theta$, $u = 2\theta = 2\tan^{-1}x$, $v = 2\tan^{-1}x \Rightarrow u = v \qquad \therefore \frac{du}{dv} = 1$

Q.140 Prove that the straight line joining the mid-points of two non-parallel sides of a trapezium is parallel to the parallel sides. **(7)**

Ans:
Let ABCD be the given trapezium. Let the position vectors of A,B,C and D with reference to some origin O be $\bar{a}, \bar{b}, \bar{c}$ *and* $\bar{d}$ respectively.

Let P and Q be the mid-points of AD and BC respectively. Then, the position vectors of P and Q are

$\frac{\bar{a} + \bar{d}}{2}$ and $\frac{\bar{b} + \bar{c}}{2}$ respectively we have , $\overline{AB} = \bar{b} - \bar{a}$ and $\overline{DC} = \bar{c} - \bar{d}$

Since $\overline{DC}$ is parallel to $\overline{AB}$, Therefore there exists a scalar λ

such that

$\overline{DC} = \bar{\lambda}\ \overline{AB} \quad \Rightarrow \quad \bar{c} - \bar{d} = \lambda(\bar{b} - \bar{a})$ -------------(1)

Now $\overline{PQ} =$ position vector of Q-position vector of P

$= \left(\frac{\bar{b}+\bar{c}}{2}\right) - \left(\frac{\bar{a}+\bar{d}}{2}\right) = \frac{1}{2}\left[(\bar{b}-\bar{a}) + (\bar{c}-\bar{d})\right] = \frac{1}{2}\left[(\bar{b}-\bar{a}) + \lambda(\bar{b}-\bar{a})\right]$

$= \frac{1}{2}(\lambda+1)(\bar{b}-\bar{a}) = \frac{1}{2}(\lambda+1)\overline{AB}$ ----------------(2)

This shows that PQ is parallel, to AB. But, AB is parallel to CD,

Therefore PQ is parallel to CD

Q.141 Find a unit vector that is perpendicular to both the vectors **(7)**

$$\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$$
$$\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$$

Ans:

$\bar{a} =$ 4i+3j+k.

$\bar{b} =$ 2i-j+2k

$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = i[6+1] - j[8-2] + k[-4-6] =$ 7i-6j-10k

$|\bar{a} \times \bar{b}| = \sqrt{49+36+100} = \sqrt{185}$

$\hat{n} = \frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|} = \frac{7i - 6j - 2k}{\sqrt{185}}$

Q.142 Find the square root of 12-6i. **(7)**

Ans:

Let z be the square root of 12-6i

then $z^2 = 12 - 6i, \quad z = x + iy$ or $x^2 - y^2 + 2ixy = 12 - 6i$

$\Rightarrow x^2 - y^2 = 12, \quad xy = -3 \quad \Rightarrow x^2 = 6 + 3\sqrt{5} \quad \Rightarrow x = \pm\sqrt{6+3\sqrt{5}}$

$y = \frac{-3}{\pm\sqrt{6+3\sqrt{5}}}$

Q.143 Evaluate the integral

$$\int \frac{3x+1}{(x-2)^2(x+2)}dx$$ **(7)**

Ans:

$$\int \frac{3x+1}{(x-2)^2(x+2)}dx$$

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2}$$

$\Rightarrow 3x+1 = A(x-2)(x+2) + B(x+2) + C(x-2)^2$ Putting $x = 2$ we get B=$\frac{7}{4}$

Putting $x = -2$ we get C=$\frac{-5}{16}$

Comparing coefficients of x^2 on both sides of the identity. we get

A+C=0$\Rightarrow A = \frac{5}{16}$ $\Rightarrow \frac{3x+1}{(x-2)^2(x+2)} = \frac{5}{16}\cdot\frac{1}{x-2} + \frac{7}{4}\cdot\frac{1}{(x-2)^2} - \frac{5}{16(x+2)}$

$$\Rightarrow \int \frac{3x+1}{(x-2)^2(x+2)}dx = \frac{5}{16}\int\frac{1}{x-2}dx + \frac{7}{4}\int\frac{dx}{(x-2)^2} - \frac{5}{16}\int\frac{dx}{x+2}$$

$$= \frac{5}{16}\log\ (x-2) - \frac{7}{4(x-2)} - \frac{5}{16}\log(x+2) + c$$

Q.144 Evaluate the definite integral

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$ **(6)**

Ans:

Let

I=$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ --------------(1)

I= $\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$

=$\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$ -------------(2)

Adding 1 and 2, we get

2I=$\int_0^{\pi} \frac{(\pi - x + x)\sin x}{1+\cos^2 x}dx = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

Let $\cos x = t, -\sin x dx = dt$

When $x = 0, t = 1$

$x = \pi, t = -1$

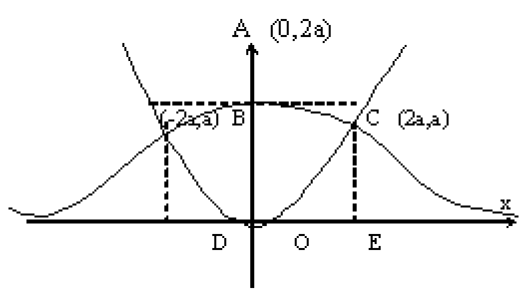
$$I=\frac{-\pi}{2}\int_{+1}^{-1}\frac{dt}{1+t^2} = \frac{-\pi}{2}\left(\tan^{-1} t\right)_{+1}^{-1} = \frac{-\pi}{2}\left[\tan^{1}(-1)-\tan^{1}(1)\right] = \frac{-\pi}{2}\left[-\frac{\pi}{4}-\frac{\pi}{4}\right] = \frac{\pi^2}{4}$$

Q.145 Find the area bounded by the parabola $x^2 = 4ay$ and the curve $y = \frac{8a^3}{x^2+4a^2}$, where a > 0. **(8)**

Ans:

The curve $y = \frac{8a^3}{x^2+4a^2}$ is symmetrical about y-axis. Equating to zero the coefficient of the highest power of x in the given equation, we find that y=0 i.e x-axis is an asymptote of the curve. Also this curve cuts the

y-axis at (0, 2a). Solving the two given equations $x^2 = 4ay$ *and* $y = \frac{8a^2}{x^2+4a^2}$ we get their points of intersection as $(\pm 2a, a)$



Now the required area OBACO

$= 2\times$ area OAC (By symmetry) $= 2\times$ [area OACE –area OCE]

$$= 2\times\left[\int_0^{2a}\frac{8a^3}{x^2+4a^2}dx - \int_0^{2a}\frac{x^2}{4a}dx\right] = 16a^3\int_0^{2a}\frac{dx}{x^2+4a^2} - \frac{1}{2a}\int_0^{2a}x^2 dx$$

$$= 16a^3\frac{1}{2a}\left[\tan^{-1}\frac{x}{2a}\right]_0^{2a} - \frac{1}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = 8a^2\times\frac{\pi}{4} - \frac{1}{2a}\times\frac{8a^3}{3}$$

$$= 2\pi a^2 - \frac{4a^2}{3} = \left[2\pi\frac{-4}{3}\right]a^2$$

Q.146 Solve the differential equation

$$\mathrm{xdy} - \mathrm{ydx} = \sqrt{\mathrm{x}^2 + \mathrm{y}^2} \quad \mathrm{dx}..$$ **(7)**

Ans:

Given

$$xdy - ydx = \sqrt{x^2 + y^2} \cdot dx$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x}$$

It is homogeneous differential equation

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2 + v^2x^2} + vx}{x}$$

$$v + x\frac{dv}{dx} = \sqrt{1 + v^2} + v$$

$$x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\log\left[v + \sqrt{1 + v^2}\right] = \log x + \log c$$

or $v + \sqrt{1 + v^2} = cx$ or $\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = cx$ or $y + \sqrt{x^2 + y^2} = cx^2$

Q.147 Find $\frac{\mathrm{dy}}{\mathrm{dx}}$**,** where $\mathrm{y} = \frac{\sqrt{\mathrm{x}^2 + 1} + \sqrt{\mathrm{x}^2 - 1}}{\sqrt{\mathrm{x}^2 + 1} - \sqrt{\mathrm{x}^2 - 1}}$ **.** **(7)**

Ans:

We have

$$y = \frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}} = \left[\frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}\right]\left[\frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}\right]$$

$$= \frac{(x^2+1)+(x^2-1)+2\sqrt{(x^2+1)(x^2-1)}}{x^2+1-x^2+1} = \frac{2x^2+2\sqrt{x^4-1}}{2}$$

$$y = x^2 + \sqrt{x^4-1}$$

$$\frac{dy}{dx} = 2x + \frac{4x^3}{2\sqrt{x^4-1}} = 2x + \frac{2x^3}{\sqrt{x^4-1}}$$

Q.148 Solve the differential equation

$$(x-y)\frac{dy}{dx} = x+3y$$ **(7)**

Ans:

$$\frac{dy}{dx} = \frac{x+3y}{x-y} \text{------------}(1)$$

Homogeneous differential equation

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

(1)becomes

$$v + x\frac{dv}{dx} = \frac{x+3vx}{x-vx} = \frac{1+3v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+3v}{1-v} - v = \frac{1+3v-v+v^2}{1-v} = \frac{1+2v+v^2}{1-v}$$

Separating the variables

$$\frac{1-v}{v^2+2v+1}dv = \frac{dx}{x}$$

on integration

$$\int \frac{1-v}{v^2+2v+1}dv = \int \frac{dx}{x}$$

$$\int \frac{1-v}{(1+v)^2}dv = \int \frac{dx}{x}$$

Let

$v+1 = t \Rightarrow v = t-1$, dv = dt

$$\int \frac{1-t+1}{t^2}dt = \int \frac{dx}{x}$$

$$2\int \frac{dt}{t^2} - \int \frac{1}{t} dt = \int \frac{dx}{x}$$

$$\frac{-2}{t} - \log t = \log cx$$

$$\log tcx = \frac{-2}{t} \Rightarrow \log cx(v+1) = \frac{-2}{v+1}$$

$$\log c(y+x) = \frac{-2x}{y+x}$$

Q.149 Two stones are thrown up from the ground simultaneously. The equation of motion for the first stone is $s = 19.6\,t - 4.9\,t^2$ and for the second stone it is $s = 9.8\,t - 4.9\,t^2$. What is the height of the second stone from the ground, when the height of the first stone is maximum. **(7)**

Ans:

$$\frac{ds}{dt} = 19.6 - 9.8t, \ \frac{ds}{dt} = 0$$

$$\Rightarrow 19.6 - 9.8t = 0$$

or $t = 2\,\text{sec}$.

Since $\frac{d^2 s}{dt^2} = -9.8 < 0$

S is maximum when t = 2sec.

Then after 2 sec. the height of the second stone from the ground is

$$S = 9.8 \times 2 - 4.9 \times 4 \quad = 19.6 - 19.6 \quad = 0$$

and the maximum height of the first stone is

$$S = 19.6 \times 2 - 4.9 \times 4 \quad = 39.2 - 19.6 \quad = 19.6$$

Q.150 Find real values of x and y if $-3 + i\,x^2 y$ and $x^2 + y + 4i$ are complex conjugate to each other. **(7)**

Ans:

Since $-3 + ix^2 y$ *and* $x^2 + y + 4i$

are complex conjugates, therefore $-3 + ix^2 y = \overline{x^2 + y + 4i}$

$\Rightarrow -3 + ix^2 y = x^2 + y - 4i \qquad \Rightarrow x^2 + y = -3 \qquad \ldots(i)$

$and \qquad x^2 y = -4 \qquad \ldots\ldots(ii) \qquad \Rightarrow -3 = x^2 - \dfrac{4}{x^2}$

$\Rightarrow x^4 + 3x^2 - 4 = 0 \qquad \Rightarrow (x^2+4)(x^2-1) = 0$

$\Rightarrow x^2 - 1 = 0 \qquad \Rightarrow x = \pm 1$

$From (ii), y = -4, when \qquad x = \pm 1$

$Hence \qquad x = 1, \qquad y = -4 \qquad and \qquad x = -1, y = -4$

Q.151 Evaluate $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+........}}}}$. **(7)**

Ans:

Let

$x = \sqrt{2+\sqrt{2+\sqrt{2+..............}}} \qquad \Rightarrow x = \sqrt{2+x}$

$\Rightarrow x^2 = 2 + x \qquad \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x^2 - 2x + x - 2 = 0 \qquad \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = -1 \qquad or \qquad x = 2$

But the given expression is positive hence $x = 2$

Q.152 Show that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1+x)^{2n-1}$. **(7)**

Ans:

Let A and B be the coefficients of x^n in the binomial expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, Then

$$A = 2n_{c_n} = \frac{2n!}{n!n!} = \frac{2n.(2n-1)!}{n(n-1)!n!} = \frac{2.(2n-1)!}{(n-1)!n!}$$

$$B = 2n-1_{c_n} = \frac{(2n-1)!}{(n-1)!n!} = \frac{1}{2}A \Rightarrow A = 2B$$

Q.153 Resolve into partial fractions $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}$, assuming a, b, c and d are distinct. **(7)**

Ans:

$$\frac{(x-a)(x-b)}{(x-c)(x-d)} = \frac{x^2-(a+b)x+ab}{x^2-(c+d)x+cd} = 1 + \frac{(c+d-a-b)x+(ab-cd)}{(x-c)(x-d)}$$

$$= 1 + \frac{(a-c)(c-b)}{d-c}.\frac{1}{x-c} + \frac{(a-d)(b-d)}{d-c}.\frac{1}{x-d}$$

Q.154 Find the general solution of the equation $\sin\theta = \sin\alpha$. **(7)**

Ans:

$\Rightarrow \sin\theta - \sin\alpha = 0$

$\Rightarrow 2\sin\left(\frac{\theta-\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) = 0$

$\Rightarrow \sin\frac{\theta-\alpha}{2} = 0 \quad Or \quad \cos\left(\frac{\theta+\alpha}{2}\right) = 0$

$\Rightarrow \frac{\theta-\alpha}{2} = m\pi \quad Or \quad \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2} \quad m \in Z$

$\Rightarrow \theta = 2m\pi + \alpha \quad or \quad \theta = (2m+1)\pi - \alpha \quad m \in Z$

$\Rightarrow \theta = (\text{any even multiple of } \pi\ ) + \alpha$

$or\ \theta = (\text{any odd multiple of } \pi\ ) - \alpha$

$\Rightarrow \theta = n\pi + (-1)^n \alpha\ ,\ n \in Z$

Q.155 If A, B and C are the angles of a triangle, show that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1. \qquad \textbf{(7)}$$

Ans:

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2}$$

$$= \tan\frac{B}{2}\left[\frac{Sin\,A/2}{Cos\,A/2} + \frac{Sin\,C/2}{Cos\,C/2}\right] + \tan C/2 \tan A/2$$

$$= \tan\frac{B}{2}\left[\frac{Sin\left(\frac{A+C}{2}\right)}{Cos\,A/2\,Cos\,C/2}\right] + \tan C/2 \tan A/2$$

$$= \frac{Sin\,B/2}{Cos\,B/2}\left[\frac{Cos\,B/2}{Cos\,A/2\,Cos\,C/2}\right] + \frac{Sin\,C/2\,Sin\,A/2}{Cos\,C/2\,Cos\,A/2}$$

$$= \frac{Sin\,B/2 + Sin\,C/2\,Sin\,A/2}{Cos\,A/2\,Cos\,C/2}$$

$$= \frac{Sin\left(\pi/2 - \frac{A+C}{2}\right) + Sin\,C/2\,Sin\,A/2}{Cos\,A/2\,Cos\,C/2}$$

$$= \frac{Cos\left(\frac{A+C}{2}\right) + Sin\,C/2\,Sin\,A/2}{Cos\,A/2\,Cos\,C/2}$$

$$=\frac{Cos\,A/2\,Cos\,C/2 - Sin\,A/2\,Sin\,C/2 + Sin\,A/2\,Sin\,C/2}{Cos\,A/2\,Cos\,C/2} = 1$$

Q.156 Find the area of a triangle whose angular points are $(K+1,1)$, $(2K+1,3)$ and $(2K+2,2K)$. Find for what value of K, these points will be collinear. **(7)**

Ans:

Here

$x_1 = k+1, \quad y_1 = 1$, $x_2 = 2k+1, \quad y_2 = 3$, $x_3 = 2k+2, \quad y_3 = 2k$

Area of Triangle

$$= \frac{1}{2}[\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}]$$

$$= \frac{1}{2}[(k+1)(3-2k)+(2k+1)(2k-1)+(2k+2)(1-3)]$$

$$= \frac{1}{2}[3k+3-2k^2-2k+4k^2-1-4k-4] = \frac{1}{2}[2k^2-3k-2]$$

Three points are collinear if Area of Triangle is zero.

$\Rightarrow 2k^2-3k-2=0 \quad \Rightarrow (2k+1)(k-2)=0 \quad \Rightarrow k=-\frac{1}{2} \quad or \quad k=2$

Q.157 If p is the length of perpendicular from the origin on a straight line whose intercepts on the axes of x and y are a and b respectively, show that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$. **(7)**

Ans:

The given line is

$\frac{x}{a}+\frac{y}{b}=1 \Rightarrow bx+ay-ab=0 \qquad \ldots(1)$

P = length of perpendicular from the origin to (1)

$$= \frac{|b(0)+a(0)-ab|}{\sqrt{b^2+a^2}} \qquad = \frac{|ab|}{\sqrt{b^2+a^2}}$$

$$P^2 = \frac{a^2b^2}{b^2+a^2} \quad \Rightarrow \frac{1}{P^2} = \frac{b^2+a^2}{a^2b^2} \quad \Rightarrow \frac{1}{P^2} = \frac{1}{a^2}+\frac{1}{b^2}$$

Q.158 Find the equation of the circle which passes through the points $(-1,2)$ and $(3,-2)$ and has its centre on the line x = 2y. **(7)**

Ans:

Let the equation of the required circles be

$x^2+y^2+2gx+2fy+c=0 \cdots (1)$

It passes through (-1,2) and (3,-2)

5-2g+4f+c=0.......................(2)
13+6g-4f+c=0.....................(3)
The centre (-g, -f) of (i) lies on x=2y
$\therefore \; -g = -2f$ or $g = 2f$ .... (4)
Solving (2), (3) and (4), we get
$g=-2,\; f=-1, c=-5$
From (1), equation is $x^2+y^2-4x-2y-5=0$

Q.159 Find the vertex, the axis, the focus and latus rectum of the parabola $y^2 = 4y + 4x$. **(7)**

Ans:
The given equation is $y^2 = 4x + 4y$
$\Rightarrow (y-2)^2 = 4(x+1)$ ....(i)
Shifting the origin to the point (-1,2) without rotating the axes and denoting the coordinates with respect to new axes by X and Y, we have
$X = x+1, Y = y-2$ ....(ii)
Using these relations in equation (i) it reduces to
$Y^2 = 4X$ ....(iii)
Here $4a = 4 \Rightarrow a = 1$
Vertex: The coordinates of vertex with new axes are X=0, Y=0
so, coordinates of the vertex with respect to old axes are (-1,2)
Focus: The coordinates of the focus w.r. to new axes are
X=1, Y=0
So, Coordinates of the focus w.r. to old axes are (0,2)
Axis: Equation of the axis of the parabola w.r. to new axes is Y=0
So, equation of axis w.r. to old axes is y=2
Latus rectum:
The length of latus rectum =4

Q.160 If $\vec{A} = i + 2j + 3k$, $\vec{B} = -i + 2j + k$ and $\vec{C} = 3i + j$, find λ such that $\vec{A} + \lambda\vec{B}$ is perpendicular to $\vec{C}$. **(7)**

Ans:
Given $\bar{A} = i + 2j + 3k$ and $\bar{B} = -i + 2j + k$
$\bar{A} + \lambda\bar{B} = (1-\lambda)i + (2+2\lambda)j + (3+\lambda)k$
Because $\bar{A} + \lambda\bar{B}$ *and* $\bar{C}$ are perpendicular $\Rightarrow \left(\bar{A} + \lambda\bar{B}\right).\bar{C} = 0$
$\Rightarrow (1-\lambda)3 + (2+2\lambda).1 = 0 \Rightarrow 3 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow 5 - \lambda = 0 \Rightarrow \lambda = 5$

Q.161 Find a unit vector normal to the plane of the vectors $\vec{A} = 3i - 2j + 4k$ and $\vec{B} = i + j - 2k$. **(7)**

Ans:

Given $\vec{A} = 3i - 2j + 4k$ and $\vec{B} = i + j - 2k$, Unit normal vector

$$\hat{n} = \frac{\pm \bar{A} \times \bar{B}}{|\bar{A} \times \bar{B}|}$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ 3 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = i[4-4] - j[-6-4] + k[3+2] = 10j + 5k$$

$$\left|\bar{A} \times \bar{B}\right| = \sqrt{100+25} = \sqrt{125}$$

$$\hat{n} = \pm\frac{10j+5k}{\sqrt{125}} = \pm\frac{2j+k}{\sqrt{5}}$$

Q.162 If $y = \tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ **.** Show that $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x4}}$. **(7)**

Ans:

Given

$$y = \tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$$

Putting $x^2 = \cos 2\theta$

$$y = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\ \theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right] = \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right] = \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^4}}$$

Q.163 Show that for all values of n, the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a}+\frac{y}{b} = 2$ at the point (a, b). **(7)**

Ans:

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

Differentiate both sides

$$n\left(\frac{x}{a}\right)^{n-1}.\frac{1}{a}+n\left(y/b\right)^{n-1}.\frac{1}{b}\frac{dy}{dx}=0\Rightarrow\left.\frac{dy}{dx}\right|_{(a,b)}=-\frac{b}{a}$$

Equation of tangent at (a,b)

$$\text{y - b}=-\frac{b}{a}(x-a)\Rightarrow bx+ay=2ab.\Rightarrow\frac{x}{a}+\frac{y}{b}=2$$

Q.164 Find the maximum and minimum values of $f(x)=(x-1)(x-2)(x-3)$. **(7)**

Ans:

$$f(x)=x^3-6x^2+11x-6 \qquad \Rightarrow\frac{df(x)}{dx}=3x^2-12x+11$$

For maxima and minima

$$\frac{df(x)}{dx}=0 \quad \Rightarrow 3x^2-12x+11=0\Rightarrow x=\frac{12\pm\sqrt{144-132}}{6} \;=\frac{12\pm\sqrt{12}}{6}$$

$$=\frac{12\pm2\sqrt{3}}{6}. \quad\Rightarrow x=\frac{6\pm\sqrt{3}}{3}$$

Again $\frac{d^2f(x)}{dx^2}=6x-12$ At $x=\frac{6+\sqrt{3}}{3}$

$$\frac{d^2f(x)}{dx^2}=6\left[\frac{6+\sqrt{3}}{3}\right]-12=12+2\sqrt{3}-12=2\sqrt{3}>0$$

At x = $\frac{6+\sqrt{3}}{3}$, $f(x)$ is minima and minimum value is $f(x)=\frac{-2\sqrt{3}}{9}$

At $x=\frac{6-\sqrt{3}}{3}$

$$\frac{d^2f(x)}{dx^2}=6\left[\frac{6-\sqrt{3}}{3}\right]-12=12-2\sqrt{3}-12=-2\sqrt{3}<0$$

At $x=\frac{6-\sqrt{3}}{3}$, f(x) is maximum and maximum value is $f(x)=\frac{2\sqrt{3}}{9}$

Q.165 Integrate the following:

(i) $\int\frac{dx}{\sqrt{x+1}-\sqrt{x}}$.

(ii) $\int\frac{x^2\tan^{-1}x}{1+x^2}dx$. **(3 + 4)**

Ans:

(i) Given $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}} \Rightarrow \int \frac{\sqrt{x+1}+\sqrt{x}}{(x+1)-x}dx = \int \sqrt{x+1}dx + \int \sqrt{x}\,\mathrm{dx}$

$= \frac{2(x+1)^{3/2}}{3} + \frac{2}{3}x^{3/2} + c$

(ii) Given

$\int \frac{x^2 \tan^{-1} x}{1+x^2}dx$

Let $\theta = \tan^{-1} x \Rightarrow x = \tan\theta,\ d\theta = \frac{1}{1+x^2}dx$

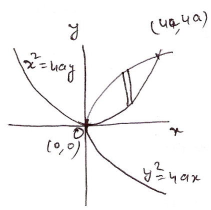
$= \int \tan^2\theta.\theta\ d\theta = \int \theta(\sec^2\theta - 1)\ d\theta = \int \theta\sec^2\theta d\theta - \int \theta d\theta$

$= \theta.\tan\theta - \int 1.\tan\theta d\theta - \frac{\theta^2}{2} = \theta\tan\theta - \int \tan\theta d\theta - \frac{\theta^2}{2}$

$= \theta\tan\theta - \log|\cos\theta| - \frac{\theta^2}{2} = x\tan^{-1} x - \frac{1}{2}\log|1+x^2| - \frac{1}{2}(\tan^{-1} x)^2 + c$

Q.166 Find the area enclosed by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. **(7)**

Ans:



The equations of the given curves are

$y^2 = 4ax$ ....(i) , $x^2 = 4ay$........................(ii)

The points of intersection of (i) and (ii) are $x = 0, y = 0$ *and* $x = 4a, y = 4a$

So, the two curves intersect at (0,0) and (4a,4a)

The region whose area we have to find is the shaded region. Here we slice this region into vertical strips. We observe that all vertical strips have lower end on the parabola $x^2 = 4ay$ and the upper end on the parabola $y^2 = 4ax$, For the approximating rectangle shown in fig, we have width Δx, *Length* $y_2 - y_1$ and the area $= (y_2 - y_1)\Delta x$

Since the approximating rectangle can move between $x = 0$ and $x = 4a$,

Thus required area $= \int_0^{4a} (y_2 - y_1)dx = \int_0^{4a} (2\sqrt{ax} - \frac{x^2}{4a})dx$

$= \left[\frac{4\sqrt{a}}{3}x^{3/2} - \frac{x^3}{12a}\right]_0^{4a} = \frac{16a^2}{3}$ sq. units.

Q.167 Find the volume of the solid of revolution obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its major axis. **(7)**

Ans:

Volume of solid $= 2\int_0^a \pi y^2 dx \quad = 2\pi b^2 \int_0^a \left(1-\frac{x^2}{a^2}\right)dx$

$= \frac{2\pi b^2}{a^2}\int_0^a (a^2-x^2)dx \quad = \frac{2\pi b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$

$= \frac{2\pi b^2}{a^2}\left[a^3-\frac{a^3}{3}\right] = \frac{2\pi b^2}{a^2}.\frac{2a^3}{3} \quad = \frac{4}{3}\pi ab^2$

Q.168 Solve the following equations :-

(i) $x\cos^2 y\,dx = y\cos^2 x\,dy$.

(ii) $\sec x\,dy + (y-\sin x)dx = 0$.

(iii) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y = x^2+\sin 2x$. **(4+5+5)**

Ans:

(i) $x\cos^2 ydx = y\cos^2 xdy$

Separating the variables

$\frac{x}{\cos^2 x}dx = \frac{y}{\cos^2 y}dy$

or $x\sec^2 xdx = y\sec^2 ydy$

Integrating both sides

$\int x\sec^2 xdx = \int y\sec^2 ydy$

or $x.\tan x - \int \tan xdx = y\tan y - \int \tan ydy + c$

or $x\tan x - \log|\cos x| = y\tan y - \log|\cos y| + c$ is the required solution.

(ii) Given

$\sec xdy + (y-\sin x)dx = 0$

$\Rightarrow \sec x\frac{dy}{dx} = -y+\sin x \Rightarrow \frac{dy}{dx}+\frac{1}{\sec x}y = \frac{\sin x}{\sec x} \Rightarrow \frac{dy}{dx}+\cos x.y = \sin x\cos x$

$I.F. = e^{\int \cos xdx} = e^{\sin x}$

Solution is

$ye^{\sin x} = \int e^{\sin x}.\sin x\cos xdx + c$

Let $\sin x = t, \cos xdx = dt$

$\int e^t.tdt = te^t - \int e^t dt \quad = te^t - e^t$

$ye^{\sin x} = (\sin x - 1)e^{\sin x} + c$

(iii) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=x^2+\sin 2x$

$A.E.$ is $m^2-4m+4=0 \quad \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

$CF=(C_1+xC_2)e^{2x}$

$$P.I.=\frac{1}{D^2-4D+4}\left(x^2+\sin 2x\right)=\frac{1}{4}\left[1-D+\frac{D^2}{4}\right]^{-1}x^2+\frac{1}{-4-4D+4}\sin 2x$$

$$=\frac{1}{4}\left[1+\left(D-\frac{D^2}{4}\right)+\left(D-\frac{D^2}{4}\right)^2+.......\right]x^2-\frac{1}{4D}\sin 2x$$

$$=\frac{1}{4}\left[x^2+2x-\frac{2}{4}+2\right]+\frac{1}{4}\frac{\cos 2x}{2}=\frac{1}{4}\left[x^2+2x+\frac{3}{2}\right]+\frac{\cos 2x}{8}$$

=

$$\Rightarrow y=C.F.+P.I.=\left(C_1+C_2x\right)e^{2x}+\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)+\frac{\cos 2x}{8}$$

is the general solution of differential equation.

Q.169 Prove that 7 divides $2^{3n}-1$ for all positive integers n. **(7)**

Ans:

Let $P(n)=2^{3n}-1$, For $n=1, P(1)=2^3-1=8-1=7$

which is divisible by 7 . Let $P(k)$ is divisible by 7 ,where k is a positive integer

i.e. $P(k)=2^{3k}-1$ is divisible by 7. We have to show that this relation is true for $n=k+1$

$P(k+1)=2^{3(k+1)}-1 \quad =2^{3k}\,2^3-1 \quad =2^{3k}\,2^3-2^3+2^3-1 \quad =2^3\,[2^{3k}-1]+7$

$P(k+1)=8(2^{3k}-1)+7$

Here $2^{3k}-1$ is divisible by 7 and 7 itself divisible by 7. Thus P(k+1) is divisible by 7. Hence result is true for $k+1$,But it is true for $n=1$. Thus it is true for every positive integer

Q.170 Find the condition that the roots of equation $ax^2+bx+c=0$ are equal. **(7)**

Ans:

Let α,β are the roots of $ax^2+bx+c=0$

$$\Rightarrow \alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Now $\alpha=\beta$

$$\Rightarrow \frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-b-\sqrt{b^2-4ac}}{2a} \quad \Rightarrow \sqrt{b^2-4ac}=-\sqrt{b^2-4ac}$$

$\Rightarrow 2\sqrt{b^2 - 4ac} = 0 \Rightarrow b^2 - 4ac = 0$

Q.171 Evaluate $\tan\left(\frac{5\pi}{12}\right)$. **(6)**

Ans:

$$\tan\left(\frac{5\pi}{12}\right) = \tan 75^\circ \quad = \tan\left(45^\circ + 30^\circ\right) \quad = \frac{\tan 45 + \tan 30}{1 - \tan 45^\circ \tan 30}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \quad = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{\left(\sqrt{3}+1\right)^2}{3-1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

Q.172 If $x + \frac{1}{x} = 2\cos\theta$, prove that $x^3 + \frac{1}{x^3} = 2\cos 3\theta$. **(8)**

Ans:

$$x + \frac{1}{x} = 2\cos\theta$$

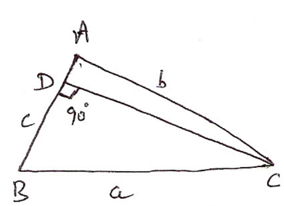
Cubic both sides

$$\left(x + \frac{1}{x}\right)^3 = 8\cos^3\theta \;\Rightarrow x^3 + \frac{1}{x^3} + 3x^2\frac{1}{x} + 3x\frac{1}{x^2} \quad = 8\cos^3\theta$$

or $x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 8\cos^3\theta \Rightarrow x^3 + \frac{1}{x^3} = \cos^3\theta - 6\cos\theta = 2\cos 3\theta$

Q.173 If a, b, c are lengths of sides opposite to angles A, B, C in a triangle ABC, then show that $a^2 = b^2 + c^2 - 2bc\cos A$. **(7)**

Ans:



When ΔABC is an acute angled triangle.Draw perpendicular CD from C on AB

In Δ CAD, we have $\cos A = \frac{AD}{b} \Rightarrow AD = b\cos A$

In ΔCBD, we have

$\cos B = \frac{BD}{a} \Rightarrow BD = a\cos B$

In ΔCBD,

$CD^2 + BD^2 = CB^2$

$CB^2 = CD^2 + (AB - AD)^2 = CD^2 + AB^2 + AD^2 - 2AB.AD$

$a^2 = AB^2 + (CD^2 + AD^2) - 2AB.AD = c^2 + AC^2 - 2AB.AD$

$a^2 = c^2 + b^2 - 2c.b\cos A$

$\Rightarrow a^2 = b^2 + c^2 - 2bc\cos A$

Q.174 Show that in a triangle ABC,
a sin (B – C) + b sin (C – A) + c sin (A – B) = 0,
where a , b, c are lengths of sides opposite to angles A, B, C. **(7)**

Ans:

Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K.$

L.H.S.

a sin(B-C)+ b sin (C-A)+c sin(A-B)

= K sin A sin(B-C)+K sin B sin(C-A)+K sin C sin(A-B)

= K[sin (B+C) sin (B-C)+sin(C+A) sin(C-A)+sin(A+B) sin(A-B)]

= K($\sin^2$ B-$\sin^2$C+ $\sin^2$C -$\sin^2$A+$\sin^2$A-$\sin^2$B]

= K(0)

= 0 = RHS

Q.175 Find the condition that the points (1, 1), (3, 5) and (a, b) are collinear. **(7)**

Ans:

Let A= (1,1) ,B= (3,5) ,C= (a,b)

The given points are collinear if $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$

$\Rightarrow 1(5-b) + 3(b-1) + a(1-5) = 0 \Rightarrow 5 - b + 3b - 3 + a - 5a = 0 \Rightarrow 2b - 4a = -2$

Or $b - 2a = -1$

Q.176 Find equations of lines which pass through the point (4, 5) and make an angle 45° with the line 2x + y +1 = 0. **(7)**

Ans:

A line through point (4,5) is $y - 5 = m(x-4)$. This makes angle 45^0 with the line

$2x + y + 1 = 0,$ whose slope is -2. Therefore.

$\tan 45^0 = \dfrac{m \pm 2}{1 \mp 2m}$ or $1 \mp 2m = m \pm 2 \Rightarrow m = -\dfrac{1}{3}, -3$

The required lines are $(y-5) = -\dfrac{1}{3}(x-4)$

$(y-5) = -3(x-4)$

Q.177 Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x - 6y - 9 = 0$ and which passes through (-4, 5). **(7)**

Ans:

Given circle is $x^2 + y^2 - 4x - 6y - 9 = 0$. Its center is (-f,-g) =(2,3)

The equation of circle whose center is (2,3) and radius r is $(x-2)^2+(y-3)^2= r^2$

It passes through (-4,5) $\Rightarrow (-4-2)^2+(5-3)^2 = r^2 \Rightarrow 36+4 = r^2 \Rightarrow r^2 = 40$

Required Circle is $(x-2)^2+(y-3)^2=40$

Q.178 Show that $y^2 - 8y - x + 19 = 0$ represents a parabola. Find its focus, vertex and directrix. **(7)**

Ans:

$y^2-8y-x+19=0 \Rightarrow (y-4)^2 = (x-3)$ ....(1)

Let Y = y-4, X= x-3 (1) becomes $Y^2=X$, which is a parabola.

Here $4a=1 \Rightarrow a = \dfrac{1}{4}$

Vertex: Vertex = (X=0, Y=0) $\Rightarrow$ (x-3=0, y-4=0) $\Rightarrow$ (x=3, y=4) So, Vertex = (3,4)

Focus: (X=a, Y=0) $\Rightarrow \left(x-3=\dfrac{1}{4}, y-4=0\right) \Rightarrow \left(x=\dfrac{13}{4}, y=4\right)$

Directrix: Equation of directorix is X= -a $\Rightarrow$ x-3= $-\dfrac{1}{4} \Rightarrow x = \dfrac{11}{4}$

Q.179 Find $\lim\limits_{x\to 0} \dfrac{\text{Sin } 3x}{x}$. **(6)**

Ans:

$$\lim_{x\to 0} \frac{Sin3x}{x} = 3\lim_{x\to 0} \frac{Sin3x}{3x} = 3.1 = 3$$

Q.180 Examine the continuity of the function f(x) = [x], where [x] is greatest integer $\le x$, x being any real number. **(8)**

Ans:

Let a be any real number, then there exists an integer k such that $k-1 \le a \le k$, Case1: a $\neq$k-1

(LHL at x=a) = $\lim\limits_{x\to a-} f(x) = \lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0}[a-h] = \lim\limits_{h\to 0}(k-1) = k-1$

(RHL at x=a) = $\lim\limits_{x\to a+} f(x) = \lim\limits_{h\to 0} f(a+h) = \lim\limits_{h\to 0}[a+h] = \lim\limits_{h\to 0}(k-1) = k-1$

and f(a)=k-1. Thus $\lim\limits_{x\to a-} f(x) = \lim\limits_{x\to a+} f(x) = f(a)$ so, f(x) is continuous at x=a.

Case2: a=k-1

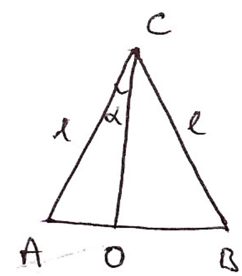
Now $\lim\limits_{x\to a-} f(x) = \lim\limits_{h\to 0}(k-1-h) = k-2$ while $\lim\limits_{x\to a+} f(x) = k-1$

Thus f(x) is not continuous at point a=k-1. Thus f(x) continuous at all points x$\neq$ an integer while it is discontinuous at integer points.

Q.181 Show that the semi verticle angle of a cone of maximum volume and a given slant height is $\tan^{-1}\sqrt{2}$. **(7)**

Ans:

Let α be the semi-vertical angle of a cone of given slant height ℓ. Then, CO=$\ell\cos\alpha$, OA=$\ell\sin\alpha$. Let V be the volume of the cone. Then



$$V = \frac{1}{3}\pi(OA)^2(CO) = \frac{1}{3}\pi\ell^3 Sin^2\alpha Cos\alpha$$

$$\frac{dV}{d\alpha} = \frac{\pi}{3}\ell^3\left[-Sin^3\alpha + 2Sin\alpha.Cos^2\alpha\right]$$

$$= \frac{\pi\ell^3}{3}Sin\alpha\left[-Sin^2\alpha + 2Cos^2\alpha\right]$$

For maximum or minimum V, $\frac{dV}{d\alpha} = 0$

$$\Rightarrow \frac{\pi\ell^3}{3}Sin\alpha\left[-Sin^2\alpha + 2Cos^2\alpha\right] = 0 \Rightarrow 2Cos^2\alpha = Sin^2\alpha$$

$or\ Sin\alpha = 0(not$ possible)

$$\Rightarrow \tan^2\alpha = 2 \Rightarrow \tan\ \alpha = \sqrt{2} \Rightarrow Cos\alpha = \frac{1}{\sqrt{3}}$$

$$Again\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi\ell^3 Cos^3\alpha\left(2 - 7\tan^2\alpha\right)$$

$$\left(\frac{d^2V}{d\alpha^2}\right)_{\tan\alpha=\sqrt{2}} = -\frac{4\pi\ell^3}{3\sqrt{3}} < 0$$

Thus V is maximum when $\tan\ \alpha = \sqrt{2} \Rightarrow \alpha = \tan^{-1}\sqrt{2}$

Q.182 Find the equation of tangent and normal to the curve $y = x^2 - 9$ at the point where it intersects the positive x-axis. **(7)**

Ans:

The equation of given curve is $y=x^2-9$.............(1)

This cuts the x-axis at the point where y=0 $\Rightarrow x^2-9=0$ $\Rightarrow x=3$

Point of contact = (3,0) Differentiating (1) w.r. to x, we get

$$\frac{dy}{dx} = 2x..............................(ii)$$

$$\left(\frac{dy}{dx}\right)_{(3,0)} = 6$$

Equation of tangent at (3,0) is y-0 =6 (x-3) $\Rightarrow$ y -6x+18=0

Evaluation of normal at (3,0) is y-0= $-\frac{1}{6}(x-3) \Rightarrow 6y + x - 3 = 0$

Q.183 Find a reduction formula for the integral $\int \sin^n x \, dx$. **(7)**

Ans:

Let $I_n = \int \sin^n x dx \quad = \int \sin^{n-1} x \sin x dx$

$$= \sin^{n-1} x(-\cos x) + \int (n-1)\sin^{n-2} x \cos x \cos x dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

$$= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = (n-1)I_{n-2} - \sin^{n-1} x \cos x, \quad n = 3,4,.........., \quad I_1 = -\cos x$$

Q.184 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$. **(7)**

Ans:

Let $I= \int_0^{\pi/2} \frac{\sqrt{Sinx}}{\sqrt{Sinx}+\sqrt{Cosx}} dx$ ………………….. (1)

$$= \int_0^{\pi/2} \frac{\sqrt{Sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{Sin\left(\frac{\pi}{2}-x\right)}+\sqrt{Cos\left(\frac{\pi}{2}-x\right)}} dx = \int_0^{\pi/2} \frac{\sqrt{Cosx}}{\sqrt{Cosx}+\sqrt{Sinx}} dx \ ......(2)$$

Now

$$2.I = \int_0^{\pi/2} \frac{\sqrt{Sinx}+\sqrt{Cosx}}{\sqrt{Sinx}+\sqrt{Cosx}} dx = \int_0^{\pi/2} dx = \pi/2 \Rightarrow I = \frac{\pi}{4}$$

Q.185 Find the area bounded by $y^2 = 4\,ax$ and its latus rectum. **(7)**

Ans:

A rough sketch of the parobola is shown in Fig.

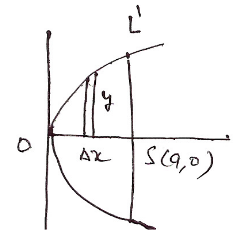
Let S(a,o) be the focus and $L'SL.$, be the latus rectum of the parabola y^2=4ax. The required area is $LOL'L.$ Since the curve is symmetric about x-axis. So, required area = 2 area ($L'OSL'$.)

Here, we slice the area $L'OSL'$ into vertical strips. For the approximating rectangle shown in fig. we have length =y, width = Δx

Area = y Δx = $\sqrt{4ax}\,\Delta x$

Since the approximating rectangle can move between x=0 and x=a

$\therefore$Required area = 2 Area $L'OSL'$ $=2\int_0^a \sqrt{4ax}\,dx = \frac{8}{3}a^2$ sq. units



Q.186 Find the volume of the solid obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, about its major axis. (7)

Ans:

$$\text{Volume of solid} = 2\int_0^a \pi y^2 dx = 2\pi b^2 \int_0^a \left(1-\frac{x^2}{a^2}\right)dx$$

$$= \frac{2\pi b^2}{a^2}\int_0^a (a^2-x^2)dx = \frac{2\pi b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[a^3-\frac{a^3}{3}\right] = \frac{2\pi b^2}{a^2}.\frac{2a^3}{3} = \frac{4}{3}\pi ab^2$$

Q.187 Solve the equation $\frac{dy}{dx}=\frac{x-y}{x+y}$. **(6)**

Ans:

$$\frac{dy}{dx}=\frac{x-y}{x+y}$$

This is homogeneous equation

Let y=vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \therefore v + x\frac{dv}{dx} = \frac{x - vx}{x + vx} \quad \Rightarrow v + x\frac{dv}{dx} = \frac{1 - v}{1 + v} \quad \Rightarrow x\frac{dv}{dx} = \frac{1 - v}{1 + v} - v$$

$$= \frac{1 - v - v - v^2}{1 + v} = \frac{1 - 2v - v^2}{1 + v}$$

Separating the variables

$$\frac{1 + v}{1 - 2v - 2v^2} dv = \frac{dx}{x}$$

Integrating both sides

$$\int \frac{1 + v}{1 - 2v - v^2} dv = \int \frac{dx}{x}. \quad \text{Let } 1\text{-}2v\text{-}v^2 = t \text{ on LHS } \Rightarrow (-2-2v)dv = dt$$

$$\Rightarrow (1+v)dv = -\frac{1}{2}dt$$

$$\therefore -\frac{1}{2}\int \frac{dt}{t} = \int \frac{dx}{x} \quad \Rightarrow -\frac{1}{2}\log t = \log x + \log c \quad \Rightarrow -\frac{1}{2}\log\left(1 - 2v - v^2\right) = \log cx$$

$$cx = (1\text{-}2v\text{-}v^2)^{-½} = \left(1 - \frac{2y}{x} - \frac{y^2}{x^2}\right)^{-1/2} = \left(\frac{x^2 - 2xy - y^2}{x^2}\right)^{-1/2}$$

$$cx = \frac{x}{\sqrt{x^2 - 2xy - y^2}} \quad or\ c\sqrt{x^2 - 2xy - y^2} = 1 \quad \Rightarrow c^2\left(x^2 - 2xy - y^2\right) = 1$$